CHAPTER 5

Introduction to Systems and Phase Plane Analysis

5.1 INTERCONNECTED FLUID TANKS

Two large tanks, each holding 24 liters of a brine solution, are interconnected by pipes as shown in Figure 5.1. Fresh water flows into tank A at a rate of 6 L/min, and fluid is drained out of tank B at the same rate; also 8 L/min of fluid are pumped from tank A to tank B, and 2 L/min from tank B to tank A. The liquids inside each tank are kept well stirred so that each mixture is homogeneous. If, initially, the brine solution in tank A contains x_0 kg of salt and that in tank B initially contains y_0 kg of salt, determine the mass of salt in each tank at time $t > 0$.†

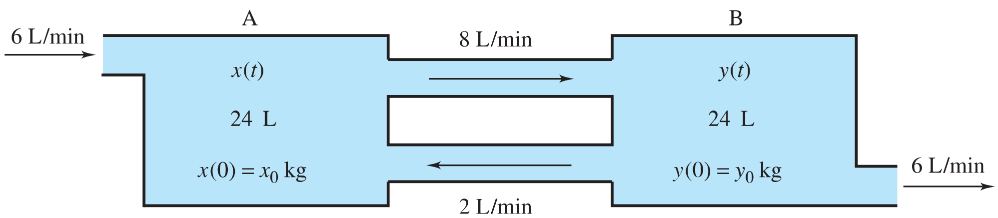

Figure 5.1 Interconnected fluid tanks

Note that the *volume* of liquid in each tank remains constant at 24 L because of the balance between the inflow and outflow volume rates. Hence, we have two unknown functions of t: the mass of salt $x(t)$ in tank A and the mass of salt $y(t)$ in tank B. By focusing attention on one tank at a time, we can derive two equations relating these unknowns. Since the system is being flushed with fresh water, we expect that the salt content of each tank will diminish to zero as $t \to +\infty$.

To formulate the equations for this system, we equate the rate of change of salt in each tank with the *net* rate at which salt is transferred to that tank. The salt *concentration* in tank A is $x(t)/24$ kg/L, so the upper interconnecting pipe carries salt out of tank A at a rate of $8x/24$ kg/min; similarly, the lower interconnecting pipe brings salt into tank A at the rate $2y/24$ kg/min (the concentration of salt in tank B is $y/24$ kg/L). The fresh water inlet, of course, transfers no salt (it simply maintains the volume in tank A at 24 L). From our premise,

$$\frac{dx}{dt} = \textbf{input rate} - \textbf{output rate} \ ,$$

†For this application we simplify the analysis by assuming the lengths and volumes of the pipes are sufficiently small that we can ignore the diffusive and advective dynamics taking place therein.

so the rate of change of the mass of salt in tank A is

$$\frac{dx}{dt} = \frac{2}{24}y - \frac{8}{24}x = \frac{1}{12}y - \frac{1}{3}x \ .$$

The rate of change of salt in tank B is determined by the same interconnecting pipes *and* by the drain pipe, carrying away $6y/24$ kg/min:

$$\frac{dy}{dt} = \frac{8}{24}x - \frac{2}{24}y - \frac{6}{24}y = \frac{1}{3}x - \frac{1}{3}y \ .$$

The interconnected tanks are thus governed by a *system* of differential equations:

$$\begin{aligned} x' &= -\frac{1}{3}x + \frac{1}{12}y \ , \\ y' &= \frac{1}{3}x - \frac{1}{3}y \ . \end{aligned} \tag{1}$$

Although both unknowns $x(t)$ and $y(t)$ appear in each of equations (1) (they are "coupled"), the structure is so transparent that we can obtain an equation for y alone by solving the second equation for x,

$$x = 3y' + y \ , \tag{2}$$

and substituting (2) in the first equation to eliminate x:

$$(3y' + y)' = -\frac{1}{3}(3y' + y) + \frac{1}{12}y \ ,$$

$$3y'' + y' = -y' - \frac{1}{3}y + \frac{1}{12}y \ ,$$

or

$$3y'' + 2y' + \frac{1}{4}y = 0 \ .$$

This last equation, which is linear with constant coefficients, is readily solved by the methods of Section 4.2. Since the auxiliary equation

$$3r^2 + 2r + \frac{1}{4} = 0$$

has roots $-1/2$, $-1/6$, a general solution is given by

$$y(t) = c_1 e^{-t/2} + c_2 e^{-t/6} \ . \tag{3}$$

Having determined y, we use equation (2) to deduce a formula for x:

$$x(t) = 3\left(-\frac{c_1}{2}e^{-t/2} - \frac{c_2}{6}e^{-t/6}\right) + c_1 e^{-t/2} + c_2 e^{-t/6} = -\frac{1}{2}c_1 e^{-t/2} + \frac{1}{2}c_2 e^{-t/6} \ . \tag{4}$$

Formulas (3) and (4) contain two undetermined parameters, c_1 and c_2, which can be adjusted to meet the specified initial conditions:

$$x(0) = -\frac{1}{2}c_1 + \frac{1}{2}c_2 = x_0 \ , \qquad y(0) = c_1 + c_2 = y_0 \ ,$$

or

$$c_1 = \frac{y_0 - 2x_0}{2}, \qquad c_2 = \frac{y_0 + 2x_0}{2}.$$

Thus, the mass of salt in tanks A and B at time t are, respectively,

$$\begin{aligned} x(t) &= -\left(\frac{y_0 - 2x_0}{4}\right)e^{-t/2} + \left(\frac{y_0 + 2x_0}{4}\right)e^{-t/6}, \\ y(t) &= \left(\frac{y_0 - 2x_0}{2}\right)e^{-t/2} + \left(\frac{y_0 + 2x_0}{2}\right)e^{-t/6}. \end{aligned} \tag{5}$$

The ad hoc elimination procedure that we used to solve this example will be generalized and formalized in the next section, to find solutions of all *linear systems with constant coefficients.* Furthermore, in later sections we will show how to extend our numerical algorithms for first-order equations to *general* systems and will consider applications to coupled oscillators and electrical systems.

It is interesting to note from (5) that all solutions of the interconnected-tanks problem tend to the constant solution $x(t) \equiv 0$, $y(t) \equiv 0$ as $t \to +\infty$. (This is of course consistent with our physical expectations.) This constant solution will be identified as a *stable equilibrium solution* in Section 5.4, in which we introduce phase plane analysis. It turns out that, for a general class of systems, equilibria can be identified and classified so as to give qualitative information about the other solutions even when we cannot solve the system explicitly.

5.2 DIFFERENTIAL OPERATORS AND THE ELIMINATION METHOD FOR SYSTEMS

The notation $y'(t) = \frac{dy}{dt} = \frac{d}{dt}y$ was devised to suggest that the derivative of a function y is the result of *operating* on the function y with the differentiation operator $\frac{d}{dt}$. Indeed, second derivatives are formed by iterating the operation: $y''(t) = \frac{d^2y}{dt^2} = \frac{d}{dt}\frac{d}{dt}y$. Commonly, the symbol D is used instead of $\frac{d}{dt}$, and the second-order differential equation

$$y'' + 4y' + 3y = 0$$

is represented[†] by

$$D^2y + 4Dy + 3y = (D^2 + 4D + 3)[y] = 0.$$

So, we have implicitly adopted the convention that the operator "product," D times D, is interpreted as the *composition* of D with itself, when it operates on functions: D^2y means $D(D[y])$; i.e., the second derivative. Similarly, the product $(D + 3)(D + 1)$ operates on a function via

$$\begin{aligned} (D+3)(D+1)[y] &= (D+3)\big[(D+1)[y]\big] = (D+3)[y' + y] \\ &= D[y' + y] + 3[y' + y] \\ &= (y'' + y') + (3y' + 3y) = y'' + 4y' + 3y = (D^2 + 4D + 3)[y]. \end{aligned}$$

[†]Some authors utilize the identity operator I, defined by $I[y] = y$, and write more formally $D^2 + 4D + 3I$ instead of $D^2 + 4D + 3$.

Thus, $(D+3)(D+1)$ is the same operator as D^2+4D+3; when they are applied to twice-differentiable functions, the results are identical.

Example 1 Show that the operator $(D+1)(D+3)$ is also the same as D^2+4D+3.

Solution For any twice-differentiable function $y(t)$, we have

$$\begin{aligned}(D+1)(D+3)[y] &= (D+1)\big[(D+3)[y]\big] = (D+1)[y'+3y]\\ &= D[y'+3y] + 1[y'+3y] = (y''+3y') + (y'+3y)\\ &= y''+4y'+3y = (D^2+4D+3)[y]\ .\end{aligned}$$

Hence, $(D+1)(D+3) = D^2+4D+3$. ◆

Since $(D+1)(D+3) = (D+3)(D+1) = D^2+4D+3$, it is tempting to generalize and propose that one can treat expressions like aD^2+bD+c as if they were ordinary polynomials in D. This is true, as long as we restrict the coefficients a, b, c to be *constants*. The following example, which has *variable* coefficients, is instructive.

Example 2 Show that $(D+3t)D$ is *not* the same as $D(D+3t)$.

Solution With $y(t)$ as before,

$$(D+3t)D[y] = (D+3t)[y'] = y''+3ty'\ ;$$

$$D(D+3t)[y] = D[y'+3ty] = y''+3y+3ty'\ .$$

They are not the same! ◆

Because the coefficient $3t$ is not a constant, it "interrupts" the interaction of the differentiation operator D with the function $y(t)$. As long as we only deal with expressions like aD^2+bD+c with *constant* coefficients $a, b,$ and $c,$ the "algebra" of differential operators follows the same rules as the algebra of polynomials. (See Problem 39 for elaboration on this point.)

This means that the familiar elimination method, used for solving *algebraic* systems like

$$\begin{aligned}3x-2y+z&=4\ ,\\ x+y-z&=0\ ,\\ 2x-y+3z&=6\ ,\end{aligned}$$

can be adapted to solve any system of *linear differential equations with constant coefficients.* In fact, we used this approach in solving the system that arose in the interconnected tanks problem of Section 5.1. Our goal in this section is to formalize this **elimination method** so that we can tackle more general linear constant coefficient systems.

We first demonstrate how the method applies to a linear system of two first-order differential equations of the form

$$a_1x'(t)+a_2x(t)+a_3y'(t)+a_4y(t) = f_1(t)\ ,$$

$$a_5x'(t)+a_6x(t)+a_7y'(t)+a_8y(t) = f_2(t)\ ,$$

where $a_1, a_2, \ldots, a_8$ are constants and $x(t), y(t)$ is the function pair to be determined. In operator notation this becomes

$$(a_1D+a_2)[x]+(a_3D+a_4)[y] = f_1\ ,$$

$$(a_5D+a_6)[x]+(a_7D+a_8)[y] = f_2\ .$$

Example 3 Solve the system

(1)
$$x'(t) = 3x(t) - 4y(t) + 1 \ ,$$
$$y'(t) = 4x(t) - 7y(t) + 10t \ .$$

Solution The alert reader may observe that since y' is absent from the first equation, we could use the latter to express y in terms of x and x' and substitute into the second equation to derive an "uncoupled" equation containing only x and its derivatives. However, this simple trick will not work on more general systems (Problem 18 is an example).

To utilize the elimination method, we first write the system using the operator notation:

(2)
$$(D - 3)[x] + 4y = 1 \ ,$$
$$-4x + (D + 7)[y] = 10t \ .$$

Imitating the elimination procedure for algebraic systems, we can eliminate x from this system by adding 4 times the first equation to $(D - 3)$ applied to the second equation. This gives

$$\big(16 + (D - 3)(D + 7)\big)[y] = 4 \cdot 1 + (D - 3)[10t] = 4 + 10 - 30t \ ,$$

which simplifies to

(3) $$(D^2 + 4D - 5)[y] = 14 - 30t \ .$$

Now equation (3) is just a second-order linear equation in y with constant coefficients that has the general solution

(4) $$y(t) = C_1 e^{-5t} + C_2 e^{t} + 6t + 2 \ ,$$

which can be found using undetermined coefficients.

To find $x(t)$, we have two options.

Method 1. We return to system (2) and eliminate y. This is accomplished by "multiplying" the first equation in (2) by $(D + 7)$ and the second equation by -4 and then adding to obtain

$$(D^2 + 4D - 5)[x] = 7 - 40t \ .$$

This equation can likewise be solved using undetermined coefficients to yield

(5) $$x(t) = K_1 e^{-5t} + K_2 e^{t} + 8t + 5 \ ,$$

where we have taken K_1 and K_2 to be the arbitrary constants, which are not necessarily the same as C_1 and C_2 used in formula (4).

It is reasonable to expect that system (1) will involve only *two* arbitrary constants, since it consists of two first-order equations. Thus, the four constants C_1, C_2, K_1, and K_2 are not independent. To determine the relationships, we substitute the expressions for $x(t)$ and $y(t)$ given in (4) and (5) into one of the equations in (1), say, the first one. This yields

$$-5K_1 e^{-5t} + K_2 e^{t} + 8 =$$
$$3K_1 e^{-5t} + 3K_2 e^{t} + 24t + 15 - 4C_1 e^{-5t} - 4C_2 e^{t} - 24t - 8 + 1 \ ,$$

which simplifies to

$$(4C_1 - 8K_1)e^{-5t} + (4C_2 - 2K_2)e^{t} = 0 \ .$$

Because e^t and e^{-5t} are linearly independent functions on any interval, this last equation holds for all t only if

$$4C_1 - 8K_1 = 0 \quad \text{and} \quad 4C_2 - 2K_2 = 0 \ .$$

Therefore, $K_1 = C_1/2$ and $K_2 = 2C_2$.

A solution to system (1) is then given by the pair

$$x(t) = \frac{1}{2}C_1e^{-5t} + 2C_2e^t + 8t + 5 \ , \qquad y(t) = C_1e^{-5t} + C_2e^t + 6t + 2 \ . \tag{6}$$

As you might expect, this pair is a **general solution** to (1) in the sense that *any* solution to (1) can be expressed in this fashion.

Method 2. A simpler method for determining $x(t)$ once $y(t)$ is known is to use the system to obtain an equation for $x(t)$ in terms of $y(t)$ and $y'(t)$. In this example we can directly solve the second equation in (1) for $x(t)$:

$$x(t) = \frac{1}{4}y'(t) + \frac{7}{4}y(t) - \frac{5}{2}t \ .$$

Substituting $y(t)$ as given in (4) yields

$$\begin{aligned} x(t) &= \frac{1}{4}\left[-5C_1e^{-5t} + C_2e^t + 6\right] + \frac{7}{4}\left[C_1e^{-5t} + C_2e^t + 6t + 2\right] - \frac{5}{2}t \\ &= \frac{1}{2}C_1e^{-5t} + 2C_2e^t + 8t + 5 \ , \end{aligned}$$

which agrees with (6). ◆

The above procedure works, more generally, for any linear system of two equations and two unknowns with *constant coefficients* regardless of the order of the equations. For example, if we let L_1, L_2, L_3, and L_4 denote linear differential operators with constant coefficients (i.e., polynomials in D), then the method can be applied to the linear system

$$\begin{aligned} L_1[x] + L_2[y] &= f_1 \ , \\ L_3[x] + L_4[y] &= f_2 \ . \end{aligned}$$

Because the system has constant coefficients, the operators commute (e.g., $L_2L_4 = L_4L_2$) and we can eliminate variables in the usual algebraic fashion. Eliminating the variable y gives

$$(L_1L_4 - L_2L_3)[x] = g_1 \ , \tag{7}$$

where $g_1 := L_4[f_1] - L_2[f_2]$. Similarly, eliminating the variable x yields

$$(L_1L_4 - L_2L_3)[y] = g_2 \ , \tag{8}$$

where $g_2 := L_1[f_2] - L_3[f_1]$. Now if $L_1L_4 - L_2L_3$ is a differential operator of order n, then a general solution for (7) contains n arbitrary constants, and a general solution for (8) also contains n arbitrary constants. Thus, a total of $2n$ constants arise. However, as we saw in Example 3, there are only n of these that are independent for the system; the remaining constants can be expressed in terms of these.† The pair of general solutions to (7) and (8) written in terms of the n independent constants is called a **general solution for the system.**

†For a proof of this fact, see *Ordinary Differential Equations*, by M. Tenenbaum and H. Pollard (Dover, New York, 1985), Chapter 7.

If it turns out that $L_1L_4 - L_2L_3$ is the zero operator, the system is said to be **degenerate.** As with the anomalous problem of solving for the points of intersection of two parallel or coincident lines, a degenerate system may have no solutions, or if it does possess solutions, they may involve any number of arbitrary constants (see Problems 23 and 24).

Elimination Procedure for 2 × 2 Systems

To find a general solution for the system

$$\begin{aligned} L_1[x] + L_2[y] &= f_1 , \\ L_3[x] + L_4[y] &= f_2 , \end{aligned}$$

where L_1, L_2, L_3, and L_4 are polynomials in $D = d/dt$:

(a) Make sure that the system is written in operator form.
(b) Eliminate one of the variables, say, y, and solve the resulting equation for $x(t)$. If the system is degenerate, stop! A separate analysis is required to determine whether or not there are solutions.
(c) (*Shortcut*) If possible, use the system to derive an equation that involves $y(t)$ but not its derivatives. [Otherwise, go to step (d).] Substitute the found expression for $x(t)$ into this equation to get a formula for $y(t)$. The expressions for $x(t)$, $y(t)$ give the desired general solution.
(d) Eliminate x from the system and solve for $y(t)$. [Solving for $y(t)$ gives more constants—in fact, twice as many as needed.]
(e) Remove the extra constants by substituting the expressions for $x(t)$ and $y(t)$ into one or both of the equations in the system. Write the expressions for $x(t)$ and $y(t)$ in terms of the remaining constants.

Example 4 Find a general solution for

$$\begin{aligned} x''(t) + y'(t) - x(t) + y(t) &= -1 , \\ x'(t) + y'(t) - x(t) &= t^2 . \end{aligned} \tag{9}$$

Solution We begin by expressing the system in operator notation:

$$\begin{aligned} (D^2 - 1)[x] + (D + 1)[y] &= -1 , \\ (D - 1)[x] + D[y] &= t^2 . \end{aligned} \tag{10}$$

Here $L_1 \coloneqq D^2 - 1$, $L_2 \coloneqq D + 1$, $L_3 \coloneqq D - 1$, and $L_4 \coloneqq D$.

Eliminating y gives [see (7)]:

$$\big((D^2 - 1)D - (D + 1)(D - 1)\big)[x] = D[-1] - (D + 1)[t^2] ,$$

which reduces to

$$(D^2 - 1)(D - 1)[x] = -2t - t^2 ,$$

$$(D - 1)^2(D + 1)[x] = -2t - t^2 . \tag{11}$$

Since $(D - 1)^2(D + 1)$ is third order, we should expect three arbitrary constants in a general solution to system (9).

Although the methods of Chapter 4 focused on solving second-order equations, we have seen several examples of how they extend in a natural way to higher-order equations.[†] Applying this strategy to the third-order equation (11), we observe that the corresponding homogeneous equation has the auxiliary equation $(r-1)^2(r+1)=0$ with roots $r=1, 1, -1$. Hence, a general solution for the homogeneous equation is

$$x_h(t) = C_1e^t + C_2te^t + C_3e^{-t} .$$

To find a particular solution to (11), we use the method of undetermined coefficients with $x_p(t) = At^2 + Bt + C$. Substituting into (11) and solving for A, B, and C yields (after a little algebra)

$$x_p(t) = -t^2 - 4t - 6 .$$

Thus, a general solution to equation (11) is

$$x(t) = x_h(t) + x_p(t) = C_1e^t + C_2te^t + C_3e^{-t} - t^2 - 4t - 6 . \tag{12}$$

To find $y(t)$, we take the shortcut described in step (c) of the elimination procedure box. Subtracting the second equation in (10) from the first, we find

$$(D^2 - D)[x] + y = -1 - t^2 ,$$

so that

$$y = (D - D^2)[x] - 1 - t^2 .$$

Inserting the expression for $x(t)$, given in (12), we obtain

$$\begin{aligned} y(t) &= C_1e^t + C_2(te^t + e^t) - C_3e^{-t} - 2t - 4 \\ &\quad -[C_1e^t + C_2(te^t + 2e^t) + C_3e^{-t} - 2] - 1 - t^2 , \end{aligned}$$

$$y(t) = -C_2e^t - 2C_3e^{-t} - t^2 - 2t - 3 . \tag{13}$$

The formulas for $x(t)$ in (12) and $y(t)$ in (13) give the desired general solution to (9). ◆

The elimination method also applies to linear systems with three or more equations and unknowns; however, the process becomes more cumbersome as the number of equations and unknowns increases. The matrix methods presented in Chapter 9 are better suited for handling larger systems. Here we illustrate the elimination technique for a 3×3 system.

Example 5 Find a general solution to

$$\begin{aligned} x'(t) &= x(t) + 2y(t) - z(t) , \\ y'(t) &= x(t) + z(t) , \\ z'(t) &= 4x(t) - 4y(t) + 5z(t) . \end{aligned} \tag{14}$$

Solution We begin by expressing the system in operator notation:

$$\begin{aligned} (D-1)[x] - 2y + z &= 0 , \\ -x + D[y] - z &= 0 , \\ -4x + 4y + (D-5)[z] &= 0 . \end{aligned} \tag{15}$$

Eliminating z from the first two equations (by adding them) and then from the last two equations yields (after some algebra, which we omit) we find

$$\begin{aligned} (D-2)[x] + (D-2)[y] &= 0 , \\ -(D-1)[x] + (D-1)(D-4)[y] &= 0 . \end{aligned} \tag{16}$$

[†]More detailed treatment of higher-order equations is given in Chapter 6.

On eliminating x from this 2×2 system, we eventually obtain

$$(D-1)(D-2)(D-3)[y] = 0 ,$$

which has the general solution

$$y(t) = C_1 e^t + C_2 e^{2t} + C_3 e^{3t} . \tag{17}$$

Taking the shortcut approach, we add the two equations in (16) to get an expression for x in terms of y and its derivatives, which simplifies to

$$x = (D^2 - 4D + 2)[y] = y'' - 4y' + 2y .$$

When we substitute the expression (17) for $y(t)$ into this equation, we find

$$x(t) = -C_1 e^t - 2C_2 e^{2t} - C_3 e^{3t} . \tag{18}$$

Finally, using the second equation in (14) to solve for $z(t)$, we get

$$z(t) = y'(t) - x(t) ,$$

and substituting in for $y(t)$ and $x(t)$ yields

$$z(t) = 2C_1 e^t + 4C_2 e^{2t} + 4C_3 e^{3t} . \tag{19}$$

The expressions for $x(t)$ in (18), $y(t)$ in (17), and $z(t)$ in (19) give a general solution with C_1, C_2, and C_3 as arbitrary constants. ◆

5.2 EXERCISES

1. Let $A = D - 1$, $B = D + 2$, $C = D^2 + D - 2$, where $D = d/dt$. For $y = t^3 - 8$, compute

(a) $A[y]$ **(b)** $B[A[y]]$ **(c)** $B[y]$
(d) $A[B[y]]$ **(e)** $C[y]$

2. Show that the operator $(D - 1)(D + 2)$ is the same as the operator $D^2 + D - 2$.

In Problems 3–18, use the elimination method to find a general solution for the given linear system, where differentiation is with respect to t.

3. $x' + 2y = 0$, $x' - y' = 0$

4. $x' = x - y$, $y' = y - 4x$

5. $x' + y' - x = 5$, $x' + y' + y = 1$

6. $x' = 3x - 2y + \sin t$, $y' = 4x - y - \cos t$

7. $(D + 1)[u] - (D + 1)[v] = e^t$, $(D - 1)[u] + (2D + 1)[v] = 5$

8. $(D - 3)[x] + (D - 1)[y] = t$, $(D + 1)[x] + (D + 4)[y] = 1$

9. $x' + y' + 2x = 0$, $x' + y' - x - y = \sin t$

10. $2x' + y' - x - y = e^{-t}$, $x' + y' + 2x + y = e^t$

11. $(D^2 - 1)[u] + 5v = e^t$, $2u + (D^2 + 2)[v] = 0$

12. $D^2[u] + D[v] = 2$, $4u + D[v] = 6$

13. $\dfrac{dx}{dt} = x - 4y$, $\dfrac{dy}{dt} = x + y$

14. $\dfrac{dx}{dt} + y = t^2$, $-x + \dfrac{dy}{dt} = 1$

15. $\dfrac{dw}{dt} = 5w + 2z + 5t$, $\dfrac{dz}{dt} = 3w + 4z + 17t$

16. $\dfrac{dx}{dt} + x + \dfrac{dy}{dt} = e^{4t}$, $2x + \dfrac{d^2y}{dt^2} = 0$

17. $x'' + 5x - 4y = 0$, $-x + y'' + 2y = 0$

18. $x'' + y'' - x' = 2t$, $x'' + y' - x + y = -1$

In Problems 19–21, solve the given initial value problem.

19. $\dfrac{dx}{dt} = 4x + y$; $x(0) = 1$, $\dfrac{dy}{dt} = -2x + y$; $y(0) = 0$

20. $\dfrac{dx}{dt} = 2x + y - e^{2t}$; $x(0) = 1$, $\dfrac{dy}{dt} = x + 2y$; $y(0) = -1$

21. $\dfrac{d^2x}{dt^2} = y \;;\quad x(0) = 3 \;,\qquad x'(0) = 1 \;,$

$\dfrac{d^2y}{dt^2} = x \;;\quad y(0) = 1 \;,\qquad y'(0) = -1$

22. Verify that the solution to the initial value problem

$$x' = 5x - 3y - 2 \;;\qquad x(0) = 2 \;,$$
$$y' = 4x - 3y - 1 \;;\qquad y(0) = 0$$

satisfies $|x(t)| + |y(t)| \to +\infty$ as $t \to +\infty$.

In Problems 23 and 24, show that the given linear system is degenerate. In attempting to solve the system, determine whether it has no solutions or infinitely many solutions.

23. $(D - 1)[x] + (D - 1)[y] = -3e^{-2t} \;,$
$(D + 2)[x] + (D + 2)[y] = 3e^{t}$

24. $D[x] + (D + 1)[y] = e^{t} \;,$
$D^2[x] + (D^2 + D)[y] = 0$

In Problems 25–28, use the elimination method to find a general solution for the given system of three equations in the three unknown functions $x(t)$, $y(t)$, $z(t)$.

25. $x' = x + 2y - z \;,$
$y' = x + z \;,$
$z' = 4x - 4y + 5z$

26. $x' = 3x + y - z \;,$
$y' = x + 2y - z \;,$
$z' = 3x + 3y - z$

27. $x' = 4x - 4z \;,$
$y' = 4y - 2z \;,$
$z' = -2x - 4y + 4z$

28. $x' = x + 2y + z \;,$
$y' = 6x - y \;,$
$z' = -x - 2y - z$

*In Problems 29 and 30, determine the range of values (if any) of the parameter λ that will ensure **all** solutions $x(t)$, $y(t)$ of the given system remain bounded as $t \to +\infty$.*

29. $\dfrac{dx}{dt} = \lambda x - y \;,$
$\dfrac{dy}{dt} = 3x + y$

30. $\dfrac{dx}{dt} = -x + \lambda y \;,$
$\dfrac{dy}{dt} = x - y$

31. Two large tanks, each holding 100 L of liquid, are interconnected by pipes, with the liquid flowing from tank A into tank B at a rate of 3 L/min and from B into A at a rate of 1 L/min (see Figure 5.2). The liquid inside each tank is kept well stirred. A brine solution with a concentration of 0.2 kg/L of salt flows into tank A at a rate of 6 L/min. The (diluted) solution flows out of the system from tank A at 4 L/min and from tank B at 2 L/min. If, initially, tank A contains pure water and tank B contains 20 kg of salt, determine the mass of salt in each tank at time $t \geq 0$.

32. In Problem 31, 3 L/min of liquid flowed from tank A into tank B and 1 L/min from B into A. Determine the mass of salt in each tank at time $t \geq 0$ if, instead, 5 L/min flows from A into B and 3 L/min flows from B into A, with all other data the same.

33. In Problem 31, assume that no solution flows out of the system from tank B, only 1 L/min flows from A into B, and only 4 L/min of brine flows into the system at tank A, other data being the same. Determine the mass of salt in each tank at time $t \geq 0$.

34. Feedback System with Pooling Delay. Many physical and biological systems involve time delays. A pure time delay has its output the same as its input but shifted in time. A more common type of delay is *pooling delay*. An example of such a feedback system is shown in Figure 5.3 on page 252. Here the level of fluid in tank B determines the rate at which fluid enters tank A. Suppose this rate is given by $R_1(t) = \alpha[V - V_2(t)]$, where α and V are positive constants and $V_2(t)$ is the volume of fluid in tank B at time t.

(a) If the outflow rate R_3 from tank B is constant and the flow rate R_2 from tank A into B is $R_2(t) = KV_1(t)$, where K is a positive constant and $V_1(t)$ is the volume of fluid in tank A at time t, then show that this feedback system is governed by the system

$$\frac{dV_1}{dt} = \alpha\big(V - V_2(t)\big) - KV_1(t) \;,$$
$$\frac{dV_2}{dt} = KV_1(t) - R_3 \;.$$

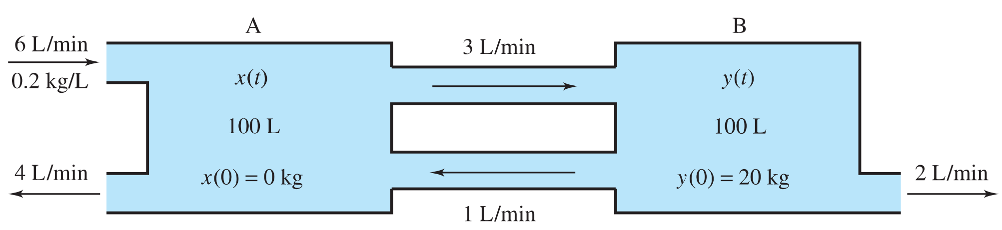

Figure 5.2 Mixing problem for interconnected tanks

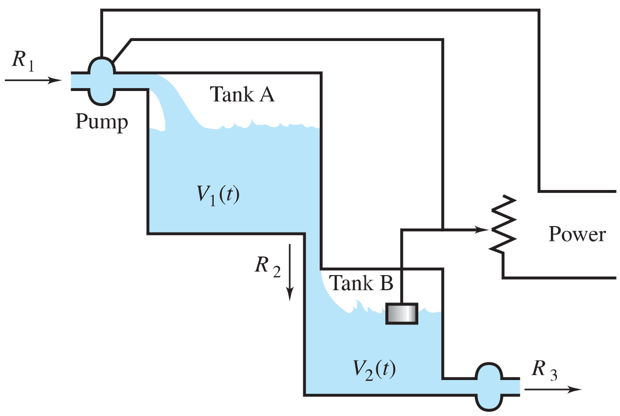

Figure 5.3 Feedback system with pooling delay

(b) Find a general solution for the system in part (a) when $\alpha = 5\ (\text{min})^{-1}$, $V = 20$ L, $K = 2\ (\text{min})^{-1}$, and $R_3 = 10$ L/min.

(c) Using the general solution obtained in part (b), what can be said about the volume of fluid in each of the tanks as $t \to +\infty$?

35. A house, for cooling purposes, consists of two zones: the attic area zone A and the living area zone B (see Figure 5.4). The living area is cooled by a 2-ton air conditioning unit that removes 24,000 Btu/hr. The heat capacity of zone B is $1/2$°F per thousand Btu. The time constant for heat transfer between zone A and the outside is 2 hr, between zone B and the outside is 4 hr, and between the two zones is 4 hr. If the outside temperature stays at 100°F, how warm does it eventually get in the attic zone A? (Heating and cooling of buildings was treated in Section 3.3.)

36. A building consists of two zones A and B (see Figure 5.5). Only zone A is heated by a furnace, which generates 80,000 Btu/hr. The heat capacity of zone A is $1/4$°F per thousand Btu. The time constant for heat transfer between zone A and the outside is 4 hr, between the unheated zone B and the outside is 5 hr, and between the two zones is 2 hr. If the outside temperature stays at 0°F, how cold does it eventually get in the unheated zone B?

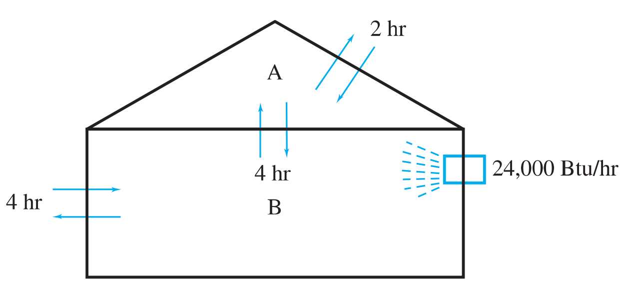

Figure 5.4 Air-conditioned house with attic

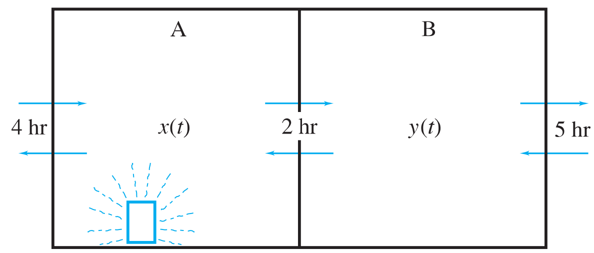

Figure 5.5 Two-zone building with one zone heated

37. In Problem 36, if a small furnace that generates 1000 Btu/hr is placed in zone B, determine the coldest it would eventually get in zone B if zone B has a heat capacity of 2°F per thousand Btu.

38. Arms Race. A simplified mathematical model for an arms race between two countries whose expenditures for defense are expressed by the variables $x(t)$ and $y(t)$ is given by the linear system

$$\frac{dx}{dt} = 2y - x + a\ ; \qquad x(0) = 1\ ,$$

$$\frac{dy}{dt} = 4x - 3y + b\ ; \qquad y(0) = 4\ ,$$

where a and b are constants that measure the trust (or distrust) each country has for the other. Determine whether there is going to be disarmament (x and y approach 0 as t increases), a stabilized arms race (x and y approach a constant as $t \to +\infty$), or a runaway arms race (x and y approach $+\infty$ as $t \to +\infty$).

39. Let A, B, and C represent three linear differential operators with constant coefficients; for example,

$$A := a_2D^2 + a_1D + a_0,\ B := b_2D^2 + b_1D + b_0,$$
$$C := c_2D^2 + c_1D + c_0,$$

where the a's, b's, and c's are constants. Verify the following properties:†

(a) Commutative laws:

$$A + B = B + A\ ,$$
$$AB = BA\ .$$

(b) Associative laws:

$$(A + B) + C = A + (B + C)\ ,$$
$$(AB)C = A(BC)\ .$$

(c) Distributive law: $A(B + C) = AB + AC$.

†We say that two operators A and B are *equal* if $A[y] = B[y]$ for all functions y with the necessary derivatives.

5.3 SOLVING SYSTEMS AND HIGHER-ORDER EQUATIONS NUMERICALLY

Although we studied a half-dozen analytic methods for obtaining solutions to first-order ordinary differential equations in Chapter 2, the techniques for higher-order equations, or systems of equations, are much more limited. Chapter 4 focused on solving the linear constant-coefficient second-order equation. The elimination method of the previous section is also restricted to constant-coefficient systems. And, indeed, higher-order linear constant-coefficient equations and systems can be solved analytically by extensions of these methods, as we will see in Chapters 6, 7, and 9.

However, if the equations—even a single second-order linear equation—have variable coefficients, the solution process is much less satisfactory. As will be seen in Chapter 8, the solutions are expressed as infinite series, and their computation can be very laborious (with the notable exception of the Cauchy–Euler, or equidimensional, equation). And we know virtually nothing about how to obtain exact solutions to nonlinear second-order equations.

Fortunately, all the cases that arise (constant or variable coefficients, nonlinear, higher-order equations or systems) can be addressed by a single formulation that lends itself to a multitude of *numerical* approaches. In this section we'll see how to express differential equations as a *system in normal form* and then show how the basic Euler method for computer solution can be easily "vectorized" to apply to such systems. Although subsequent chapters will return to analytic solution methods, the vectorized version of the Euler technique or the more efficient Runge–Kutta technique will hereafter be available as fallback methods for numerical exploration of intractable problems.

Normal Form

A system of m differential equations in the m unknown functions $x_1(t), x_2(t), \ldots, x_m(t)$ expressed as

$$\text{(1)} \qquad \begin{aligned} x_1'(t) &= f_1(t, x_1, x_2, \ldots, x_m) \ , \\ x_2'(t) &= f_2(t, x_1, x_2, \ldots, x_m) \ , \\ &\vdots \\ x_m'(t) &= f_m(t, x_1, x_2, \ldots, x_m) \end{aligned}$$

is said to be in **normal form.** Notice that (1) consists of m first-order equations that collectively look like a *vectorized* version of the single generic first-order equation

$$\text{(2)} \qquad x' = f(t, x) \ ,$$

and that the system expressed in equation (1) of Section 5.1 takes this form, as do equations (1) and (14) in Section 5.2. An initial value problem for (1) entails finding a solution to this system that satisfies the initial conditions

$$x_1(t_0) = a_1, \quad x_2(t_0) = a_2, \quad \ldots, \quad x_m(t_0) = a_m$$

for prescribed values $t_0, a_1, a_2, \ldots, a_m$.

The importance of the normal form is underscored by the fact that most professional codes for initial value problems presume that the system is written in this form. Furthermore, for a *linear* system in normal form, the powerful machinery of linear algebra can be readily applied. [Indeed, in Chapter 9 we will show how the solutions $x(t) = ce^{at}$ of the simple equation $x' = ax$ can be generalized to constant-coefficient systems in normal form.]

For these reasons it is gratifying to note that a (single) higher-order equation can always be converted to an equivalent system of first-order equations.

To convert an mth-order differential equation

$$y^{(m)}(t) = f\left(t, y, y', \ldots, y^{(m-1)}\right) \tag{3}$$

into a first-order system, we introduce, as additional unknowns, the sequence of derivatives of y:

$$x_1(t) := y(t), \quad x_2(t) := y'(t), \quad \ldots, \quad x_m(t) := y^{(m-1)}(t) .$$

With this scheme, we obtain $m - 1$ first-order equations quite trivially:

$$\begin{aligned} x_1'(t) &= y'(t) = x_2(t) , \\ x_2'(t) &= y''(t) = x_3(t) , \\ &\vdots \\ x_{m-1}'(t) &= y^{(m-1)}(t) = x_m(t) . \end{aligned} \tag{4}$$

The mth and final equation then constitutes a restatement of the original equation (3) in terms of the new unknowns:

$$x_m'(t) = y^{(m)}(t) = f(t, x_1, x_2, \ldots, x_m) . \tag{5}$$

If equation (3) has initial conditions $y(t_0) = a_1, y'(t_0) = a_2, \ldots, y^{(m-1)}(t_0) = a_m$, then the system (4)–(5) has initial conditions $x_1(t_0) = a_1, x_2(t_0) = a_2, \ldots, x_m(t_0) = a_m$.

Example 1 Convert the initial value problem

$$y''(t) + 3ty'(t) + y(t)^2 = \sin t ; \qquad y(0) = 1, \quad y'(0) = 5 \tag{6}$$

into an initial value problem for a system in normal form.

Solution We first express the differential equation in (6) as

$$y''(t) = -3ty'(t) - y(t)^2 + \sin t .$$

Setting $x_1(t) := y(t)$ and $x_2(t) := y'(t)$, we obtain

$$\begin{aligned} x_1'(t) &= x_2(t) , \\ x_2'(t) &= -3tx_2(t) - x_1(t)^2 + \sin t . \end{aligned}$$

The initial conditions transform to $x_1(0) = 1, x_2(0) = 5$. ◆

Euler's Method for Systems in Normal Form

Recall from Section 1.4 that Euler's method for solving a single first-order equation (2) is based on estimating the solution x at time $(t_0 + h)$ using the approximation

$$x(t_0 + h) \approx x(t_0) + hx'(t_0) = x(t_0) + hf\left(t_0, x(t_0)\right) , \tag{7}$$

and that as a consequence the algorithm can be summarized by the recursive formulas

$$t_{n+1} = t_n + h , \tag{8}$$

$$x_{n+1} = x_n + hf(t_n, x_n), \qquad n = 0, 1, 2, \ldots \tag{9}$$

[compare equations (2) and (3), Section 1.4]. Now we can apply the approximation (7) to each of the equations in the system (1):

$$x_k(t_0 + h) \approx x_k(t_0) + hx_k'(t_0) = x_k(t_0) + hf_k\left(t_0, x_1(t_0), x_2(t_0), \ldots, x_m(t_0)\right) , \tag{10}$$

and for $k = 1, 2, \ldots m$, we are led to the recursive formulas

(11) $$t_{n+1} = t_n + h \ ,$$

(12) $$\begin{aligned} x_{1;n+1} &= x_{1;n} + hf_1(t_n, x_{1;n}, x_{2;n}, \ldots, x_{m;n}) \ , \\ x_{2;n+1} &= x_{2;n} + hf_2(t_n, x_{1;n}, x_{2;n}, \ldots, x_{m;n}) \ , \\ &\vdots \\ x_{m;n+1} &= x_{m;n} + hf_m(t_n, x_{1;n}, x_{2;n}, \ldots, x_{m;n}) \qquad (n = 0, 1, 2, \ldots) \ . \end{aligned}$$

Here we are burdened with the ungainly notation $x_{p;n}$ for the approximation to the value of the pth-function x_p at time $t = t_0 + nh$; i.e., $x_{p;n} \approx x_p(t_0 + nh)$. However, if we treat the unknowns and right-hand members of (1) as components of vectors

$$\begin{aligned} \mathbf{x}(t) &:= [x_1(t), x_2(t), \ldots, x_m(t)] \ , \\ \mathbf{f}(t, \mathbf{x}) &= [f_1(t, x_1, x_2, \ldots, x_m), f_2(t, x_1, x_2, \ldots, x_m), \ldots, f_m(t, x_1, x_2, \ldots, x_m)] \ , \end{aligned}$$

then (12) can be expressed in the much neater form

(13) $$\mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}(t_n, \mathbf{x}_n) \ .$$

Example 2 Use the vectorized Euler method with step size $h = 0.1$ to find an approximation for the solution to the initial value problem

(14) $$y''(t) + 4y'(t) + 3y(t) = 0; \qquad y(0) = 1.5 \ , \quad y'(0) = -2.5 \ ,$$

on the interval [0, 1].

Solution For the given step size, the method will yield approximations for $y(0.1), y(0.2), \ldots, y(1.0)$. To apply the vectorized Euler method to (14), we first convert it to normal form. Setting $x_1 = y$ and $x_2 = y'$, we obtain the system

(15) $$\begin{aligned} x_1' &= x_2; & x_1(0) &= 1.5 \ , \\ x_2' &= -4x_2 - 3x_1; & x_2(0) &= -2.5 \ . \end{aligned}$$

Comparing (15) with (1) we see that $f_1(t, x_1, x_2) = x_2$ and $f_2(t, x_1, x_2) = -4x_2 - 3x_1$. With the starting values of $t_0 = 0$, $x_{1;0} = 1.5$, and $x_{2;0} = -2.5$, we compute

$$\begin{cases} x_1(0.1) \approx x_{1;1} = x_{1;0} + hx_{2;0} = 1.5 + 0.1(-2.5) = 1.25 \ , \\ x_2(0.1) \approx x_{2;1} = x_{2;0} + h(-4x_{2;0} - 3x_{1;0}) = -2.5 + 0.1[-4(-2.5) - 3 \cdot 1.5] = -1.95 \ ; \end{cases}$$

$$\begin{cases} x_1(0.2) \approx x_{1;2} = x_{1;1} + hx_{2;1} = 1.25 + 0.1(-1.95) = 1.055 \ , \\ x_2(0.2) \approx x_{2;2} = x_{2;1} + h(-4x_{2;1} - 3x_{1;1}) = -1.95 + 0.1[-4(-1.95) - 3 \cdot 1.25] = -1.545 \ . \end{cases}$$

Continuing the algorithm we compute the remaining values. These are listed in Table 5.1 on page 256, along with the exact values calculated via the methods of Chapter 4. Note that the $x_{2;n}$ column gives approximations to $y'(t)$, since $x_2(t) \equiv y'(t)$. ◆

TABLE 5.1 Approximations of the Solution to (14) in Example 2

$t = n(0.1)$	$x_{1;n}$	y Exact	$x_{2;n}$	y' Exact
0	1.5	1.5	−2.5	−2.5
0.1	1.25	1.275246528	−1.95	−2.016064749
0.2	1.055	1.093136571	−1.545	−1.641948207
0.3	0.9005	0.944103051	−1.2435	−1.35067271
0.4	0.77615	0.820917152	−1.01625	−1.122111364
0.5	0.674525	0.71809574	−0.842595	−0.9412259
0.6	0.5902655	0.63146108	−0.7079145	−0.796759968
0.7	0.51947405	0.557813518	−0.60182835	−0.680269946
0.8	0.459291215	0.494687941	−0.516939225	−0.585405894
0.9	0.407597293	0.440172416	−0.4479509	−0.507377929
1	0.362802203	0.392772975	−0.391049727	−0.442560044

Euler's method is modestly accurate for this problem with a step size of $h = 0.1$. The next example demonstrates the effects of using a sequence of smaller values of h to improve the accuracy.

Example 3 For the initial value problem of Example 2, use Euler's method to estimate $y(1)$ for successively halved step sizes $h = 0.1, 0.05, 0.025, 0.0125, 0.00625$.

Solution Using the same scheme as in Example 2, we find the following approximations, denoted by $y(1;h)$ (obtained with step size h):

h	0.1	0.05	0.025	0.0125	0.00625
$y(1;h)$	0.36280	0.37787	0.38535	0.38907	0.39092

[Recall that the exact value, rounded to 5 decimal places, is $y(1) = 0.39277$.] ◆

The Runge–Kutta scheme described in Section 3.7 is easy to vectorize also; details are given on the following page. As would be expected, its performance is considerably more accurate, yielding five-decimal agreement with the exact solution for a step size of 0.05:

h	0.1	0.05	0.025	0.0125	0.00625
$y(1;h)$	0.39278	0.39277	0.39277	0.39277	0.39277

As in Section 3.7, both algorithms can be coded so as to repeat the calculation of $y(1)$ with a sequence of smaller step sizes until two consecutive estimates agree to within some prespecified tolerance ε. Here one should interpret "two estimates agree to within ε" to mean that *each component* of the successive vector approximants [i.e., approximants to $y(1)$ and $y'(1)$] should agree to within ε.

An Application to Population Dynamics

A mathematical model for the population dynamics of competing species, one a predator with population $x_2(t)$ and the other its prey with population $x_1(t)$, was developed independently in the early 1900s by A. J. Lotka and V. Volterra. It assumes that there is plenty of food available for the prey to eat, so the birthrate of the prey should follow the Malthusian or exponential law (see Section 3.2); that is, the birthrate of the prey is Ax_1, where A is a positive constant. The death rate of the prey depends on the number of interactions between the predators and the prey. This is modeled by the expression Bx_1x_2, where B is a positive constant. Therefore, the rate of change in the population of the prey per unit time is $dx_1/dt = Ax_1 - Bx_1x_2$. Assuming that the predators depend entirely on the prey for their food, it is argued that the birthrate of the predators depends on the number of interactions with the prey; that is, the birthrate of predators is Dx_1x_2, where D is a positive constant. The death rate of the predators is assumed to be Cx_2 because without food the population would die off at a rate proportional to the population present. Hence, the rate of change in the population of predators per unit time is $dx_2/dt = -Cx_2 + Dx_1x_2$. Combining these two equations, we obtain the Volterra–Lotka system for the population dynamics of two competing species:

$$
\begin{aligned}
x_1' &= Ax_1 - Bx_1x_2 \ , \\
x_2' &= -Cx_2 + Dx_1x_2 \ .
\end{aligned}
\tag{16}
$$

Such systems are in general not explicitly solvable. In the following example, we obtain an approximate solution for such a system by utilizing the vectorized form of the Runge–Kutta algorithm.

For the system of two equations

$$
\begin{aligned}
x_1' &= f_1(t, x_1, x_2) \ , \\
x_2' &= f_2(t, x_1, x_2) \ ,
\end{aligned}
$$

with initial conditions $x_1(t_0) = x_{1;0}$, $x_2(t_0) = x_{2;0}$, the vectorized form of the Runge–Kutta recursive equations (cf. (14), page 134) becomes

$$
\begin{cases}
t_{n+1} := t_n + h \quad (n = 0, 1, 2, \ldots) \ , \\
x_{1;n+1} := x_{1;n} + \frac{1}{6}(k_{1,1} + 2k_{1,2} + 2k_{1,3} + k_{1,4}) \ , \\
x_{2;n+1} := x_{2;n} + \frac{1}{6}(k_{2,1} + 2k_{2,2} + 2k_{2,3} + k_{2,4}) \ ,
\end{cases}
\tag{17}
$$

where h is the step size and, for $i = 1$ and 2,

$$
\begin{cases}
k_{i,1} := hf_i(t_n, x_{1;n}, x_{2;n}) \ , \\
k_{i,2} := hf_i\left(t_n + \frac{h}{2},\ x_{1;n} + \frac{1}{2}k_{1,1},\ x_{2;n} + \frac{1}{2}k_{2,1}\right) \ , \\
k_{i,3} := hf_i\left(t_n + \frac{h}{2},\ x_{1;n} + \frac{1}{2}k_{1,2},\ x_{2;n} + \frac{1}{2}k_{2,2}\right) \ , \\
k_{i,4} := hf_i(t_n + h,\ x_{1;n} + k_{1,3},\ x_{2;n} + k_{2,3}) \ .
\end{cases}
\tag{18}
$$

It is important to note that both $k_{1,1}$ and $k_{2,1}$ must be computed before either $k_{1,2}$ or $k_{2,2}$. Similarly, both $k_{1,2}$ and $k_{2,2}$ are needed to compute $k_{1,3}$ and $k_{2,3}$, etc. In Appendix F, program outlines are given for applying the method to graph approximate solutions over a specified interval $[t_0, t_1]$ or to obtain approximations of the solutions at a specified point to within a desired tolerance.

Example 4 Use the classical fourth-order Runge–Kutta algorithm for systems to approximate the solution of the initial value problem

$$\text{(19)} \qquad \begin{aligned} x_1' &= 2x_1 - 2x_1x_2 \ ; & x_1(0) &= 1 \ , \\ x_2' &= x_1x_2 - x_2 \ ; & x_2(0) &= 3 \end{aligned}$$

at $t = 1$. Starting with $h = 1$, continue halving the step size until two successive approximations of $x_1(1)$ and of $x_2(1)$ differ by at most 0.0001.

Solution Here $f_1(t, x_1, x_2) = 2x_1 - 2x_1x_2$ and $f_2(t, x_1, x_2) = x_1x_2 - x_2$. With the inputs $t_0 = 0$, $x_{1;0} = 1$, $x_{2;0} = 3$, we proceed with the algorithm to compute $x_1(1; 1)$ and $x_2(1; 1)$, the approximations to $x_1(1)$, $x_2(1)$ using $h = 1$. We find from the formulas in (18) that

$$\begin{aligned}
k_{1,1} &= h(2x_{1;0} - 2x_{1;0}x_{2;0}) = 2(1) - 2(1)(3) = -4 \ , \\
k_{2,1} &= h(x_{1;0}x_{2;0} - x_{2;0}) = (1)(3) - 3 = 0 \ , \\
k_{1,2} &= h\left[2\left(x_{1;0} + \tfrac{1}{2}k_{1,1}\right) - 2\left(x_{1;0} + \tfrac{1}{2}k_{1,1}\right)\left(x_{2;0} + \tfrac{1}{2}k_{2,1}\right)\right] \\
&= 2\left[1 + \tfrac{1}{2}(-4)\right] - 2\left[1 + \tfrac{1}{2}(-4)\right]\left[3 + \tfrac{1}{2}(0)\right] \\
&= -2 + 2(3) = 4 \ , \\
k_{2,2} &= h\left[\left(x_{1;0} + \tfrac{1}{2}k_{1,1}\right)\left(x_{2;0} + \tfrac{1}{2}k_{2,1}\right) - \left(x_{2;0} + \tfrac{1}{2}k_{2,1}\right)\right] \\
&= \left[1 + \tfrac{1}{2}(-4)\right]\left[3 + \tfrac{1}{2}(0)\right] - \left[3 + \tfrac{1}{2}(0)\right] \\
&= (-1)(3) - 3 = -6 \ ,
\end{aligned}$$

and similarly we compute

$$\begin{aligned}
k_{1,3} &= h\left[2\left(x_{1;0} + \tfrac{1}{2}k_{1,2}\right) - 2\left(x_{1;0} + \tfrac{1}{2}k_{1,2}\right)\left(x_{2;0} + \tfrac{1}{2}k_{2,2}\right)\right] = 6 \ , \\
k_{2,3} &= h\left[\left(x_{1;0} + \tfrac{1}{2}k_{1;2}\right)\left(x_{2;0} + \tfrac{1}{2}k_{2,2}\right) - \left(x_{2;0} + \tfrac{1}{2}k_{2,2}\right)\right] = 0 \ , \\
k_{1,4} &= h\left[2(x_{1;0} + k_{1,3}) - 2(x_{1;0} + k_{1,3})(x_{2;0} + k_{2,3})\right] = -28 \ , \\
k_{2,4} &= h\left[(x_{1;0} + k_{1,3})(x_{2;0} + k_{2,3}) - (x_{2;0} + k_{2,3})\right] = -18 \ .
\end{aligned}$$

Inserting these values into formula (17), we get

$$\begin{aligned}
x_{1;1} &= x_{1;0} + \frac{1}{6}(k_{1,1} + 2k_{1,2} + 2k_{1,3} + k_{1,4}) \\
&= 1 + \frac{1}{6}(-4 + 8 + 12 - 28) = -1 \ , \\
x_{2;1} &= x_{2;0} + \frac{1}{6}(k_{2,1} + 2k_{2,2} + 2k_{2,3} + k_{2,4}) \ , \\
&= 3 + \frac{1}{6}(0 - 12 + 0 + 18) = 4 \ ,
\end{aligned}$$

as the respective approximations to $x_1(1)$ and $x_2(1)$.

Repeating the algorithm with $h = 1/2$ $(N = 2)$ we obtain the approximations $x_1(1; 2^{-1})$ and $x_2(1; 2^{-1})$ for $x_1(1)$ and $x_2(1)$. In Table 5.2, we list the approximations $x_1(1; 2^{-m})$ and $x_2(1; 2^{-m})$ for $x_1(1)$ and $x_2(1)$ using step size $h = 2^{-m}$ for $m = 0, 1, 2, 3$, and 4. We stopped at $m = 4$, since both

$$|x_1(1; 2^{-3}) - x_1(1; 2^{-4})| = 0.00006 < 0.0001$$

and

$$|x_2(1; 2^{-3}) - x_2(1; 2^{-4})| = 0.00001 < 0.0001 \ .$$

Hence, $x_1(1) \approx 0.07735$ and $x_2(1) \approx 1.46445$, with tolerance 0.0001. ◆

TABLE 5.2 Approximations of the Solution to System (19) in Example 4

m	h	$x_1(1; h)$	$x_2(1; h)$
0	1.0	−1.0	4.0
1	0.5	0.14662	1.47356
2	0.25	0.07885	1.46469
3	0.125	0.07741	1.46446
4	0.0625	0.07735	1.46445

To get a better feel for the solution to system (19), we have graphed in Figure 5.6 an approximation of the solution for $0 \le t \le 12$, using linear interpolation to connect the vectorized Runge–Kutta approximants for the points $t = 0, 0.125, 0.25, \ldots, 12.0$ (i.e., with $h = 0.125$). From the graph it appears that the components x_1 and x_2 are periodic in the variable t. Phase plane analysis is used in Section 5.5 to show that, indeed, Volterra–Lotka equations have periodic solutions.

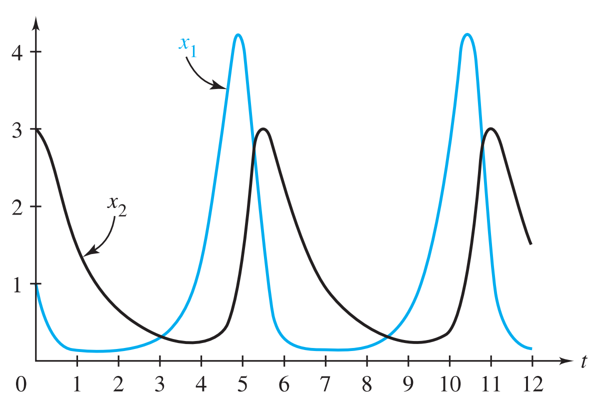

Figure 5.6 Graphs of the components of an approximate solution to the Volterra–Lotka system (17)

5.3 EXERCISES

In Problems 1–7, convert the given initial value problem into an initial value problem for a system in normal form.

1. $y''(t) + ty'(t) - 3y(t) = t^2$;
$y(0) = 3$, $y'(0) = -6$

2. $y''(t) = \cos(t - y) + y^2(t)$;
$y(0) = 1$, $y'(0) = 0$

3. $y^{(4)}(t) - y^{(3)}(t) + 7y(t) = \cos t$;
$y(0) = y'(0) = 1$, $y''(0) = 0$, $y^{(3)}(0) = 2$

4. $y^{(6)}(t) = [y'(t)]^3 - \sin\big(y(t)\big) + e^{2t}$;
$y(0) = y'(0) = \cdots = y^{(5)}(0) = 0$

5. $x'' + y - x' = 2t$; $x(3) = 5$, $x'(3) = 2$,
$y'' - x + y = -1$; $y(3) = 1$, $y'(3) = -1$
[*Hint:* Set $x_1 = x$, $x_2 = x'$, $x_3 = y$, $x_4 = y'$.]

6. $3x'' + 5x - 2y = 0$; $x(0) = -1$, $x'(0) = 0$,
$4y'' + 2y - 6x = 0$; $y(0) = 1$, $y'(0) = 2$

7. $x''' - y = t$; $x(0) = x'(0) = x''(0) = 4$,
$2x'' + 5y'' - 2y = 1$; $y(0) = y'(0) = 1$

8. Sturm–Liouville Form. A second-order equation is said to be in **Sturm–Liouville form** if it is expressed as
$$[p(t)y'(t)]' + q(t)y(t) = 0 \ .$$
Show that the substitutions $x_1 = y, x_2 = py'$ result in the normal form
$$x_1' = x_2/p \ ,$$
$$x_2' = -qx_1 \ .$$
If $y(0) = a$ and $y'(0) = b$ are the initial values for the Sturm–Liouville problem, what are $x_1(0)$ and $x_2(0)$?

9. In Section 3.6, we discussed the improved Euler's method for approximating the solution to a first-order equation. Extend this method to normal systems and give the recursive formulas for solving the initial value problem.

In Problems 10–13, use the vectorized Euler method with $h = 0.25$ to find an approximation for the solution to the given initial value problem on the specified interval.

10. $y'' + ty' + y = 0$;
$y(0) = 1$, $y'(0) = 0$ on $[0, 1]$

11. $(1 + t^2)y'' + y' - y = 0$;
$y(0) = 1$, $y'(0) = -1$ on $[0, 1]$

12. $t^2y'' + y = t + 2$;
$y(1) = 1$, $y'(1) = -1$ on $[1, 2]$

13. $y'' = t^2 - y^2$;
$y(0) = 0$, $y'(0) = 1$ on $[0, 1]$
(Can you guess the solution?)

In Problems 14–24, you will need a computer and a programmed version of the vectorized classical fourth-order Runge–Kutta algorithm. (At the instructor's discretion, other algorithms may be used.)†

14. Using the vectorized Runge–Kutta algorithm with $h = 0.5$, approximate the solution to the initial value problem
$$3t^2y'' - 5ty' + 5y = 0 \ ;$$
$$y(1) = 0 \ , \qquad y'(1) = \frac{2}{3}$$
at $t = 8$. Compare this approximation to the actual solution $y(t) = t^{5/3} - t$.

15. Using the vectorized Runge–Kutta algorithm, approximate the solution to the initial value problem
$$y'' = t^2 + y^2 \ ; \qquad y(0) = 1 \ , \qquad y'(0) = 0$$
at $t = 1$. Starting with $h = 1$, continue halving the step size until two successive approximations [of both $y(1)$ and $y'(1)$] differ by at most 0.01.

16. Using the vectorized Runge–Kutta algorithm for systems with $h = 0.125$, approximate the solution to the initial value problem
$$x' = 2x - y \ ; \qquad x(0) = 0 \ ,$$
$$y' = 3x + 6y \ ; \qquad y(0) = -2$$
at $t = 1$. Compare this approximation to the actual solution
$$x(t) = e^{5t} - e^{3t} \ , \qquad y(t) = e^{3t} - 3e^{5t} \ .$$

17. Using the vectorized Runge–Kutta algorithm, approximate the solution to the initial value problem
$$\frac{du}{dx} = 3u - 4v \ ; \qquad u(0) = 1 \ ,$$
$$\frac{dv}{dx} = 2u - 3v \ ; \qquad v(0) = 1$$

†An applet, maintained on the Web at http://alamos.math.arizona.edu/~rychlik/JOde/index.html, automates most of the differential equation algorithms discussed in this book.

at $x = 1$. Starting with $h = 1$, continue halving the step size until two successive approximations of $u(1)$ and $v(1)$ differ by at most 0.001.

18. Combat Model. A simplified mathematical model for conventional versus guerrilla combat is given by the system

$$x_1' = -(0.1)x_1x_2 \ ; \qquad x_1(0) = 10 \ ,$$
$$x_2' = -x_1 \ ; \qquad x_2(0) = 15 \ ,$$

where x_1 and x_2 are the strengths of guerrilla and conventional troops, respectively, and 0.1 and 1 are the *combat effectiveness coefficients.* Who will win the conflict: the conventional troops or the guerrillas? [*Hint:* Use the vectorized Runge–Kutta algorithm for systems with $h = 0.1$ to approximate the solutions.]

19. Predator–Prey Model. The Volterra–Lotka predator–prey model predicts some rather interesting behavior that is evident in certain biological systems. For example, suppose you fix the initial population of prey but increase the initial population of predators. Then the population cycle for the prey becomes more severe in the sense that there is a long period of time with a reduced population of prey followed by a short period when the population of prey is very large. To demonstrate this behavior, use the vectorized Runge–Kutta algorithm for systems with $h = 0.5$ to approximate the populations of prey x and of predators y over the period $[0, 5]$ that satisfy the Volterra–Lotka system

$$x' = x(3 - y) \ ,$$
$$y' = y(x - 3)$$

under each of the following initial conditions:

(a) $x(0) = 2 \ , \quad y(0) = 4 \ .$
(b) $x(0) = 2 \ , \quad y(0) = 5 \ .$
(c) $x(0) = 2 \ , \quad y(0) = 7 \ .$

20. In Group Project C of Chapter 4, it was shown that the simple pendulum equation

$$\theta''(t) + \sin\theta(t) = 0$$

has periodic solutions when the initial displacement and velocity are small. Show that the period of the solution may depend on the initial conditions by using the vectorized Runge–Kutta algorithm with $h = 0.02$ to approximate the solutions to the simple pendulum problem on $[0, 4]$ for the initial conditions:

(a) $\theta(0) = 0.1 \ , \qquad \theta'(0) = 0 \ .$
(b) $\theta(0) = 0.5 \ , \qquad \theta'(0) = 0 \ .$
(c) $\theta(0) = 1.0 \ , \qquad \theta'(0) = 0 \ .$

[*Hint:* Approximate the length of time it takes to reach $-\theta(0)$.]

21. Fluid Ejection. In the design of a sewage treatment plant, the following equation arises:†

$$60 - H = (77.7)H'' + (19.42)(H')^2 \ ;$$
$$H(0) = H'(0) = 0 \ ,$$

where H is the level of the fluid in an ejection chamber and t is the time in seconds. Use the vectorized Runge–Kutta algorithm with $h = 0.5$ to approximate $H(t)$ over the interval $[0, 5]$.

22. Oscillations and Nonlinear Equations. For the initial value problem

$$x'' + (0.1)(1 - x^2)x' + x = 0 \ ;$$
$$x(0) = x_0 \ , \qquad x'(0) = 0 \ ,$$

use the vectorized Runge–Kutta algorithm with $h = 0.02$ to illustrate that as t increases from 0 to 20, the solution x exhibits damped oscillations when $x_0 = 1$, whereas x exhibits expanding oscillations when $x_0 = 2.1$.

23. Nonlinear Spring. The **Duffing equation**

$$y'' + y + ry^3 = 0 \ ,$$

where r is a constant, is a model for the vibrations of a mass attached to a *nonlinear* spring. For this model, does the period of vibration vary as the parameter r is varied? Does the period vary as the initial conditions are varied? [*Hint:* Use the vectorized Runge–Kutta algorithm with $h = 0.1$ to approximate the solutions for $r = 1$ and 2, with initial conditions $y(0) = a$, $y'(0) = 0$ for $a = 1$, 2, and 3.]

24. Pendulum with Varying Length. A pendulum is formed by a mass m attached to the end of a wire that is attached to the ceiling. Assume that the length $l(t)$ of the wire varies with time in some predetermined fashion. If $\theta(t)$ is the

†See *Numerical Solution of Differential Equations,* by William Milne (Dover, New York, 1970), p. 82.

angle between the pendulum and the vertical, then the motion of the pendulum is governed by the initial value problem

$$l^2(t)\theta''(t) + 2l(t)l'(t)\theta'(t) + gl(t)\theta(t) = 0 \ ;$$
$$\theta(0) = \theta_0 \ , \qquad \theta'(0) = \theta_1 \ ,$$

where g is the acceleration due to gravity. Assume that

$$l(t) = l_0 + l_1 \cos(\omega t - \phi) \ ,$$

where l_1 is much smaller than l_0. (This might be a model for a person on a swing, where the *pumping* action changes the distance from the center of mass of the swing to the point where the swing is attached.) To simplify the computations, take $g = 1$. Using the Runge–Kutta algorithm with $h = 0.1$, study the motion of the pendulum when $\theta_0 = 0.5$, $\theta_1 = 0$, $l_0 = 1$, $l_1 = 0.1$, $\omega = 1$, and $\phi = 0.02$. In particular, does the pendulum ever attain an angle greater in absolute value than the initial angle θ_0? Does the total arc traversed during one-half of a swing ever exceed 1?

In Problems 25–30, use a software package or the SUBROUTINE in Appendix F.

25. Using the Runge–Kutta algorithm for systems with $h = 0.05$, approximate the solution to the initial value problem

$$y''' + y'' + y^2 = t \ ;$$
$$y(0) = 1 \ , \qquad y'(0) = 0 \ , \qquad y''(0) = 1$$

at $t = 1$.

26. Use the Runge–Kutta algorithm for systems with $h = 0.1$ to approximate the solution to the initial value problem

$$\begin{aligned} x' &= yz \ ; & x(0) &= 0 \ , \\ y' &= -xz \ ; & y(0) &= 1 \ , \\ z' &= -xy/2 \ ; & z(0) &= 1 \ , \end{aligned}$$

at $t = 1$.

27. Generalized Blasius Equation. H. Blasius, in his study of laminar flow of a fluid, encountered an equation of the form

$$y''' + yy'' = (y')^2 - 1 \ .$$

Use the Runge–Kutta algorithm for systems with $h = 0.1$ to approximate the solution that satisfies the initial conditions $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 1.32824$. Sketch this solution on the interval $[0, 2]$.

28. Lunar Orbit. The motion of a moon moving in a planar orbit about a planet is governed by the equations

$$\frac{d^2x}{dt^2} = -G\frac{mx}{r^3} \ , \qquad \frac{d^2y}{dt^2} = -G\frac{my}{r^3} \ ,$$

where $r := (x^2 + y^2)^{1/2}$, G is the gravitational constant, and m is the mass of the moon. Assume $Gm = 1$. When $x(0) = 1$, $x'(0) = y(0) = 0$, and $y'(0) = 1$, the motion is a circular orbit of radius 1 and period 2π.

(a) Setting $x_1 = x, \ x_2 = x', \ x_3 = y, \ x_4 = y'$, express the governing equations as a first-order system in normal form.

(b) Using $h = 2\pi/100 \approx 0.0628318$, compute one orbit of this moon (i.e., do $N = 100$ steps?). Do your approximations agree with the fact that the orbit is a circle of radius 1?

29. Competing Species. Let $p_i(t)$ denote, respectively, the populations of three competing species S_i, $i = 1, 2, 3$. Suppose these species have the same growth rates, and the maximum population that the habitat can support is the same for each species. (We assume it to be one unit.) Also suppose the competitive advantage that S_1 has over S_2 is the same as that of S_2 over S_3 and S_3 over S_1. This situation is modeled by the system

$$\begin{aligned} p_1' &= p_1(1 - p_1 - ap_2 - bp_3) \ , \\ p_2' &= p_2(1 - bp_1 - p_2 - ap_3) \ , \\ p_3' &= p_3(1 - ap_1 - bp_2 - p_3) \ , \end{aligned}$$

where a and b are positive constants. To demonstrate the population dynamics of this system when $a = b = 0.5$, use the Runge–Kutta algorithm for systems with $h = 0.1$ to approximate the populations p_i over the time interval $[0, 10]$ under each of the following initial conditions:

(a) $p_1(0) = 1.0 \ , \quad p_2(0) = 0.1 \ , \quad p_3(0) = 0.1$.
(b) $p_1(0) = 0.1 \ , \quad p_2(0) = 1.0 \ , \quad p_3(0) = 0.1$.
(c) $p_1(0) = 0.1 \ , \quad p_2(0) = 0.1 \ , \quad p_3(0) = 1.0$.

On the basis of the results of parts (a)–(c), decide what you think will happen to these populations as $t \to +\infty$.

30. Spring Pendulum. Let a mass be attached to one end of a spring with spring constant k and the other end attached to the ceiling. Let l_0 be the natural length of the spring and let $l(t)$ be its length at time t. If $\theta(t)$ is the angle between the pendulum and the vertical, then the motion of the spring pendulum is governed by the system

$$l''(t) - l(t)\theta'(t) - g\cos\theta(t) + \frac{k}{m}(l - l_0) = 0 \ ,$$

$$l^2(t)\theta''(t) + 2l(t)l'(t)\theta'(t) + gl(t)\sin\theta(t) = 0 \ .$$

Assume $g = 1$, $k = m = 1$, and $l_0 = 4$. When the system is at rest, $l = l_0 + mg/k = 5$.

(a) Describe the motion of the pendulum when $l(0) = 5.5$, $l'(0) = 0$, $\theta(0) = 0$, and $\theta'(0) = 0$.

(b) When the pendulum is both stretched and given an angular displacement, the motion of the pendulum is more complicated. Using the Runge–Kutta algorithm for systems with $h = 0.1$ to approximate the solution, sketch the graphs of the length l and the angular displacement θ on the interval $[0, 10]$ if $l(0) = 5.5$, $l'(0) = 0$, $\theta(0) = 0.5$, and $\theta'(0) = 0$.

5.4 INTRODUCTION TO THE PHASE PLANE

In this section, we study systems of two first-order equations of the form

$$\textbf{(1)} \qquad \frac{dx}{dt} = f(x, y) \ , \qquad \frac{dy}{dt} = g(x, y) \ .$$

Note that the independent variable t does not appear in the right-hand terms $f(x, y)$ and $g(x, y)$; such systems are called **autonomous.** For example, the system that modeled the interconnected tanks problem in Section 5.1,

$$x' = -\frac{1}{3}x + \frac{1}{12}y \ ,$$
$$y' = \frac{1}{3}x - \frac{1}{3}y \ ,$$

is autonomous. So is the Volterra–Lotka system,

$$x' = Ax - Bxy \ ,$$
$$y' = -Cy + Dxy \ ,$$

(with A, B, C, D constants), which was discussed in Example 4 of Section 5.3 as a model for population dynamics.

For future reference, we note that the solutions to autonomous systems have a "time-shift immunity," in the sense that if the pair $x(t)$, $y(t)$ solves (1), so does the **time-shifted** pair $x(t + c)$, $y(t + c)$ for any constant c. Specifically, if we let $X(t) \coloneqq x(t + c)$ and $Y(t) \coloneqq y(t + c)$, then by the chain rule

$$\frac{dX}{dt}(t) = \frac{dx}{dt}(t + c) = f\big(x(t + c), y(t + c)\big) = f\big(X(t), Y(t)\big) \ ,$$
$$\frac{dY}{dt}(t) = \frac{dy}{dt}(t + c) = g\big(x(t + c), y(t + c)\big) = g\big(X(t), Y(t)\big) \ ,$$

proving that $X(t)$, $Y(t)$ is also a solution to (1).

Since t does not appear explicitly in the system (1), it is certainly tempting to divide the two equations, invoke the chain rule

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$

and consider the single first-order differential equation

$$\textbf{(2)} \qquad \frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}.$$

We will refer to (2) as the **phase plane equation.** In Chapters 1 and 2 we mastered several approaches to equations like (2): the use of *direction fields* to visualize the solution graphs, and the analytic techniques for the cases of separability, linearity, exactness, etc.

So the form (2) certainly has advantages over (1), but it is important to maintain our perspective by noting these distinctions:

(i) A solution to the original problem (1) is a *pair* of functions of t—namely, $x(t)$ and $y(t)$—that satisfies (1) for all t in some interval I. These functions can be visualized as a pair of graphs, as in Figure 5.7. If, in the xy-plane, we plot the points $(x(t), y(t))$ as t varies over I, the resulting curve is known as the **trajectory** of the solution pair $x(t)$, $y(t)$, and the xy-plane is called the **phase plane** in this context (see Figure 5.8 on page 265). Note, however, that the trajectory in this plane contains less information than the original graphs, because the t-dependence has been suppressed. (Typically, though, we indicate the *direction* of time with an arrow on the curve.) In principle we can construct, point by point, the trajectory from the solution graphs, but we cannot reconstruct the solution graphs from the phase plane trajectory alone (because we would not know what value of t to assign to each point).

(ii) Nonetheless, the slope dy/dx of a trajectory in the phase plane is given by the right-hand side of (2). So, in solving equation (2) we are indeed locating the *trajectories* of the system (1) in the phase plane. More precisely, we have shown that the trajectories satisfy equation (2), and thus lie on its solution curves.

(iii) In Chapters 1 and 2, we regarded x as the independent variable and y as the dependent variable, in equations of the form (2). This is no longer true in the context of the system (1); x and y are both dependent variables on an equal footing, and t is the independent variable.

Thus, it appears that a phase plane portrait may be a useful, albeit incomplete, tool for analyzing first-order autonomous systems like (1).

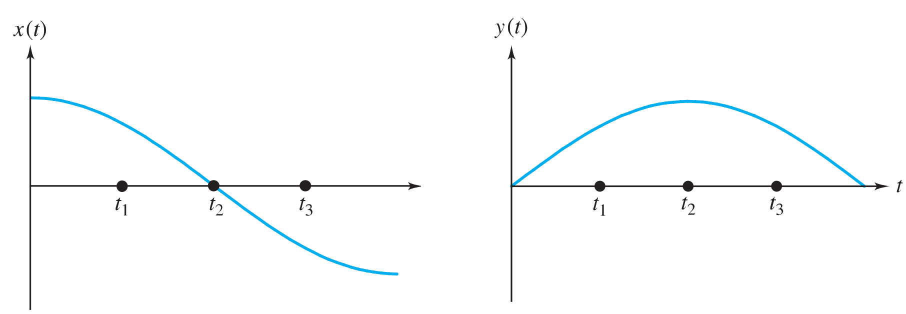

Figure 5.7 Solution pair for system (1)

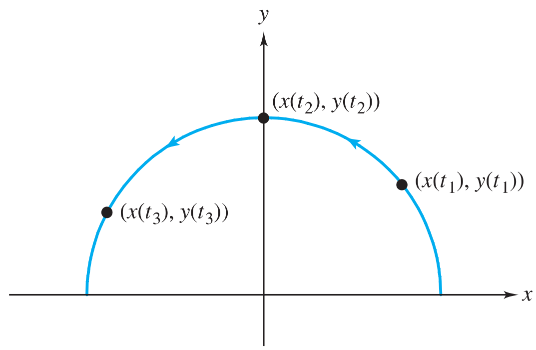

Figure 5.8 Phase plane trajectory of the solution pair for system (1)

Except for the very special case of linear systems with constant coefficients that was discussed in Section 5.2, finding all solutions to the system (1) is generally an impossible task. But it is relatively easy to find *constant solutions;* if $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$, then the constant functions $x(t) \equiv x_0$, $y(t) \equiv y_0$ solve (1). For such solutions the following terminology is used.

Critical Points and Equilibrium Solutions

Definition 1. A point (x_0, y_0) where $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$ is called a **critical point,** or **equilibrium point,** of the system $dx/dt = f(x, y)$, $dy/dt = g(x, y)$, and the corresponding constant solution $x(t) \equiv x_0$, $y(t) \equiv y_0$ is called an **equilibrium solution.** The set of all critical points is called the **critical point set.**

Notice that trajectories of equilibrium solutions consist of just single points (the equilibrium points). But what can be said about the other trajectories? Can we predict any of their features from closer examination of the equilibrium points? To explore this we focus on the phase plane equation (2) and exploit its direction field (recall Section 1.3, page 15). However, we'll augment the direction field plot by attaching arrowheads to the line segments, indicating the direction of the "flow" of solutions as t increases. This is easy: When dx/dt is positive, $x(t)$ increases so the trajectory flows to the right. Therefore, according to (1), all direction field segments drawn in a region where $f(x, y)$ is positive should point to the right [and, of course, they point to the left if $f(x, y)$ is negative]. If $f(x, y)$ is zero, we can use $g(x, y)$ to decide if the flow is upward [$y(t)$ increases] or downward [$y(t)$ decreases]. [*What if both $f(x, y)$ and $g(x, y)$ are zero?*]

In the examples that follow, one can use computers or calculators for generating these direction fields.

Example 1 Sketch the direction field in the phase plane for the system

$$\textbf{(3)}\qquad \begin{aligned} \frac{dx}{dt} &= -x\ , \\ \frac{dy}{dt} &= -2y \end{aligned}$$

and identify its critical point.

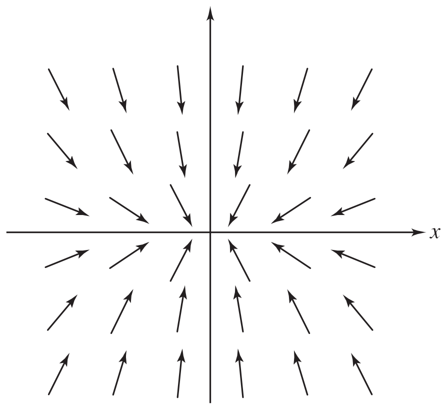

Figure 5.9 Direction field for Example 1

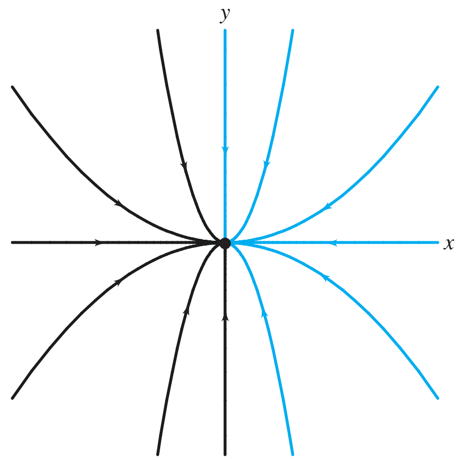

Figure 5.10 Trajectories for Example 1

Solution Here $f(x, y) = -x$ and $g(x, y) = -2y$ are both zero when $x = y = 0$, so $(0, 0)$ is the critical point. The direction field for the phase plane equation

$$\frac{dy}{dx} = \frac{-2y}{-x} = \frac{2y}{x} \tag{4}$$

is given in Figure 5.9. Since $dx/dt = -x$ in (3), trajectories in the right half-plane (where $x > 0$) flow to the left, and vice versa. From the figure we can see that all solutions "flow into" the critical point $(0, 0)$. Such a critical point is called **asymptotically stable.**† ◆

Remark. For this simple example, we can actually solve the system (3) explicitly; indeed, (3) constitutes an uncoupled pair of linear equations whose solutions are $x(t) = c_1e^{-t}$ and $y(t) = c_2e^{-2t}$. By elimination of t, we obtain the equation $y = c_2e^{-2t} = c_2[x(t)/c_1]^2 = cx^2$. So the trajectories lie along the parabolas $y = cx^2$. [Alternatively, we could have separated variables in (4) and identified these parabolas as the phase plane solution curves.] Notice that each such parabola is made up of three trajectories: an incoming trajectory approaching the origin in the right half-plane; its mirror-image trajectory approaching the origin in the left half-plane; and the origin itself, an equilibrium point. Sample trajectories are indicated in Figure 5.10.

Example 2 Sketch the direction field in the phase plane for the system

$$\begin{aligned} \frac{dx}{dt} &= x \ , \\ \frac{dy}{dt} &= 2y \end{aligned} \tag{5}$$

and describe the behavior of solutions near the critical point $(0, 0)$.

Solution This example is almost identical to the previous one; in fact, one could say we have merely "reversed time" in (3). The direction field segments for

$$\frac{dy}{dx} = \frac{2y}{x} \tag{6}$$

are the same as those of (4), but the direction arrows are reversed. Now all solutions flow *away* from the critical point $(0, 0)$; the equilibrium is **unstable.** ◆

†See Section 12.3 for a rigorous exposition of stability and critical points. All references to Chapters 11–13 refer to the expanded text *Fundamentals of Differential Equations and Boundary Value Problems,* 6th ed.

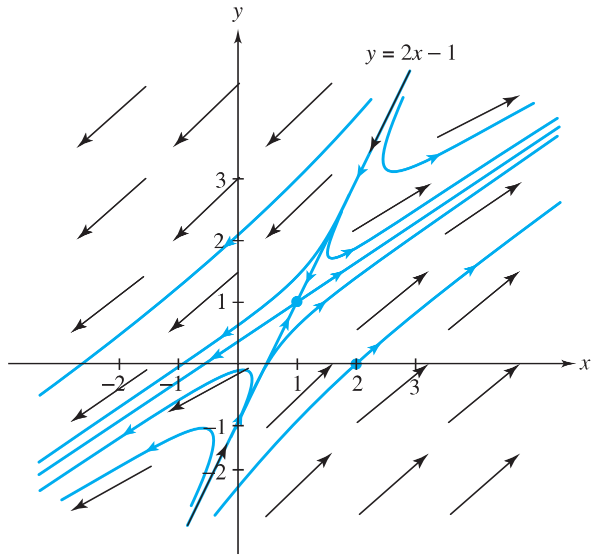

Figure 5.11 Direction field and trajectories for Example 3

Example 3 For the system (7) below, find the critical points, sketch the direction field in the phase plane, and predict the asymptotic nature (i.e., behavior as $t \to +\infty$) of the solution starting at $x = 2$, $y = 0$ when $t = 0$.

$$\frac{dx}{dt} = 5x - 3y - 2 \ ,$$
$$\frac{dy}{dt} = 4x - 3y - 1 \ . \tag{7}$$

Solution The only critical point is the solution of the simultaneous equations $f(x, y) = g(x, y) = 0$:

$$5x_0 - 3y_0 - 2 = 0 \ ,$$
$$4x_0 - 3y_0 - 1 = 0 \ , \tag{8}$$

from which we find $x_0 = y_0 = 1$. The direction field for the phase plane equation

$$\frac{dy}{dx} = \frac{4x - 3y - 1}{5x - 3y - 2} \tag{9}$$

is shown in Figure 5.11, with some trajectories rough-sketched in by hand.[†] Note that solutions flow to the right for $5x - 3y - 2 > 0$, i.e., for all points *below* the line $5x - 3y - 2 = 0$.

The phase plane solution curve passing through $(2, 0)$ in Figure 5.11 apparently extends to infinity. Does this imply the corresponding *system* solution $x(t)$, $y(t)$ also approaches infinity in the sense that $|x(t)| + |y(t)| \to +\infty$ as $t \to +\infty$, or could its trajectory "stall" at some point along the phase plane solution curve, or possibly even "backtrack"? *It cannot backtrack,* because the direction arrows along the trajectory point, unambiguously, to the right. And if $\big(x(t), y(t)\big)$ stalls at some point (x_1, y_1), then intuitively we would conclude that (x_1, y_1) was an equilibrium point (since the "speeds" dx/dt and dy/dt would approach zero there). But we have already found the only critical point. So we conclude, with a high degree of confidence,[††] that the system solution does indeed go to infinity.

[†] The phase plane solution curves could be obtained analytically by solving equation (9) using the methods of Section 2.6.

[††] These informal arguments are made more rigorous in Chapter 12. All references to Chapters 11–13 refer to the expanded text *Fundamentals of Differential Equations and Boundary Value Problems,* 6th edition.

The critical point $(1, 1)$ is **unstable** because, although many solutions get arbitrarily close to $(1, 1)$, most of them eventually flow away. Solutions that lie on the line $y = 2x - 1$, however, do converge to $(1, 1)$. Such an equilibrium is an example of a **saddle point.** ◆

In the preceding example, we informally argued that if a trajectory "stalls"—that is, if it has an endpoint—then this endpoint would have to be a critical point. This is more carefully stated in the following theorem, whose proof is outlined in Problem 30.

Endpoints Are Critical Points

Theorem 1. Let the pair $x(t)$, $y(t)$ be a solution on $[0, +\infty)$ to the autonomous system $dx/dt = f(x, y)$, $dy/dt = g(x, y)$, where f and g are continuous in the plane. If the limits

$$x^* := \lim_{t \to +\infty} x(t) \qquad \text{and} \qquad y^* := \lim_{t \to +\infty} y(t)$$

exist and are finite, then the point (x^*, y^*) is a critical point for the system.

Some typical trajectory configurations near critical points are displayed and classified in Figure 5.12. These phase plane portraits arise from the systems listed in Problem 29, and can be generated by software packages having trajectory-sketching options[†]. A more complete discussion of the nature of various types of equilibrium solutions and their stability is deferred to

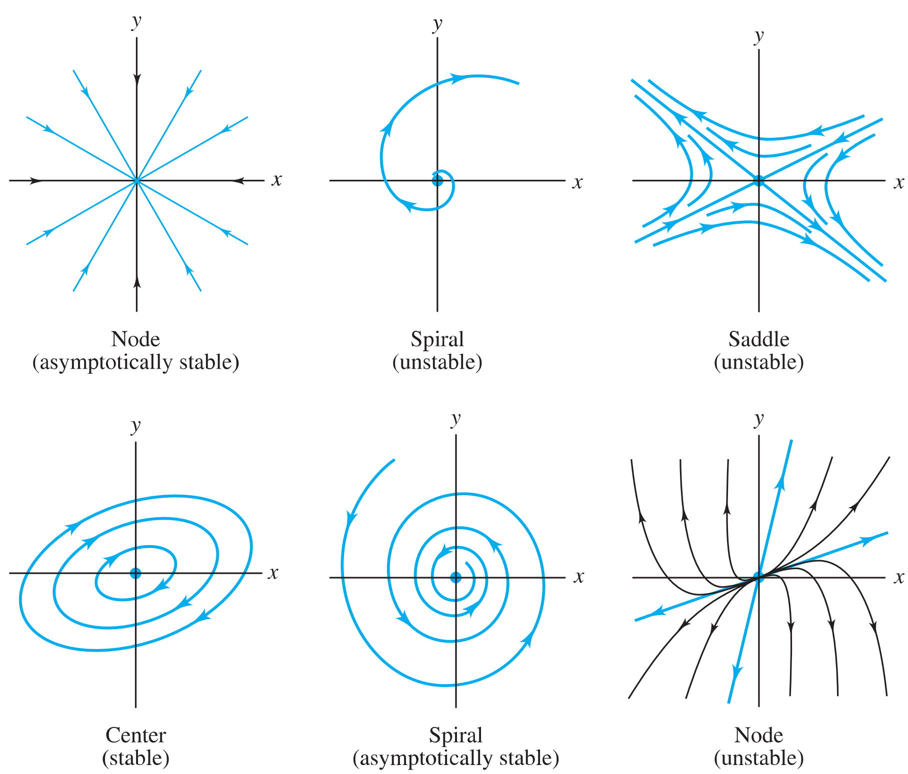

Figure 5.12 Examples of different trajectory behaviors near critical point at origin

[†]An applet, maintained on the Web at http://alamos.math.arizona.edu/~rychlik/JOde/index.html, automates most of the differential equation algorithms discussed in this book.

Chapter 12.[†] For the moment, however, notice that unstable critical points are distinguished by "runaway" trajectories emanating from arbitrarily nearby points, while stable equilibria "trap" all neighboring trajectories. The asymptotically stable critical points *attract* their neighboring trajectories as $t \to +\infty$.

Historically, the phase plane was introduced to facilitate the analysis of mechanical systems governed by Newton's second law, force equals mass times acceleration. An autonomous mechanical system arises when this force is independent of time and can be modeled by a second-order equation of the form

$$y'' = f(y, y') \ . \tag{10}$$

As we have seen in Section 5.3, this equation can be converted to a normal first-order system by introducing the velocity $v = dy/dt$ and writing

$$\begin{aligned} \frac{dy}{dt} &= v \ , \\ \frac{dv}{dt} &= f(y, v) \ . \end{aligned} \tag{11}$$

Thus, we can analyze the behavior of an autonomous mechanical system by studying its phase plane diagram in the yv-plane. Notice that with v as the vertical axis, trajectories $\left(y(t), v(t)\right)$ flow to the right in the upper half-plane (where $v > 0$), and to the left in the lower half-plane.

Example 4 Sketch the direction field in the phase plane for the first-order system corresponding to the unforced, undamped mass–spring oscillator described in Section 4.1 (Figure 4.1, page 153). Sketch several trajectories and interpret them physically.

Solution The equation derived in Section 4.1 for this oscillator is $my'' + ky = 0$ or, equivalently, $y'' = -ky/m$. Hence, the system (11) takes the form

$$\begin{aligned} y' &= v \ , \\ v' &= -\frac{ky}{m} \ . \end{aligned} \tag{12}$$

The critical point is at the origin $y = v = 0$. The direction field in Figure 5.13 on page 270 indicates that the trajectories appear to be either closed curves (ellipses?) or spirals that encircle the critical point.

We saw in Section 4.9 that the undamped oscillator motions are periodic; they cycle repeatedly through the same sets of points, with the same velocities. Their trajectories in the phase plane, then, must be closed curves.[††] Let's confirm this mathematically by solving the phase plane equation

$$\frac{dv}{dy} = -\frac{ky}{mv} \ . \tag{13}$$

Equation (13) is separable, and we find

$$v\,dv = -\frac{ky}{m}dy \qquad \text{or} \qquad d\left(\frac{v^2}{2}\right) = -\frac{k}{m}d\left(\frac{y^2}{2}\right) \ ,$$

[†] All references to Chapters 11–13 refer to the expanded text *Fundamentals of Differential Equations and Boundary Value Problems,* 6th edition.

[††] By the same reasoning, *underdamped* oscillations would correspond to spiral trajectories asymptotically approaching the origin as $t \to +\infty$.

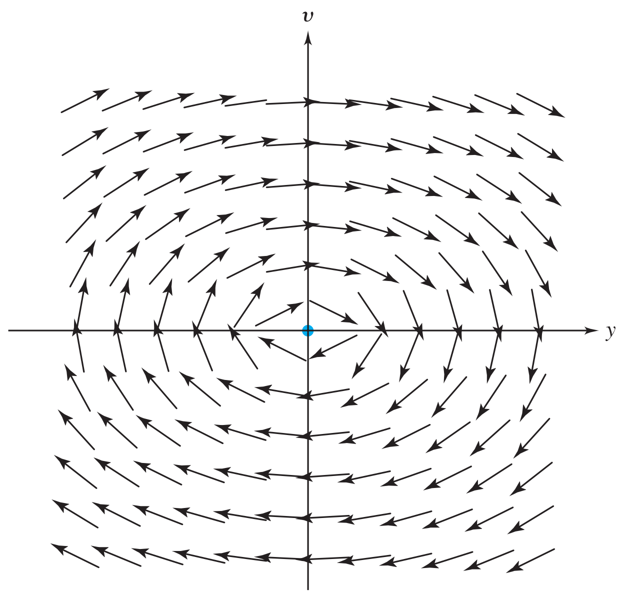

Figure 5.13 Direction field for Example 4

so its solutions are the ellipses $v^2/2 + ky^2/2m = C$ as shown in Figure 5.14. The solutions of (12) are confined to these ellipses and hence flow neither toward nor away from the equilibrium solution. The critical point is thus identified as a **center** in Figure 5.12 on page 268.

Furthermore, the system solutions must continually circulate around the ellipses, since there are no critical points to stop them. This confirms that all solutions are periodic. ◆

Remark. More generally, we argue that if a solution to an autonomous system like (1) passes through a point in the phase plane *twice* and if it is sufficiently well behaved to satisfy a *uniqueness theorem,* then the second "tour" satisfies the same initial conditions as the first tour and so must replicate it. In other words, *closed trajectories containing no critical points correspond to periodic solutions.*

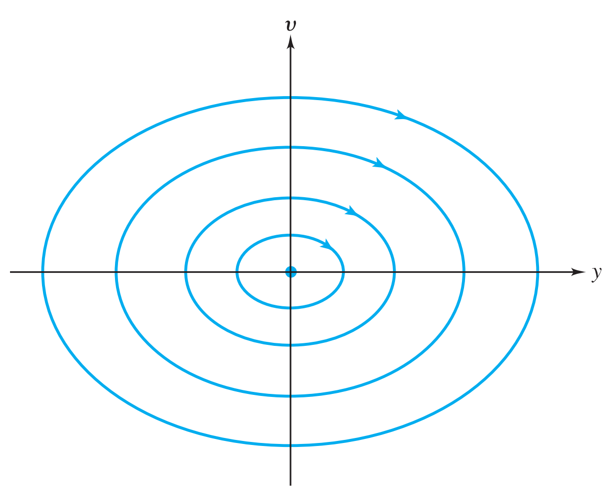

Figure 5.14 Trajectories for Example 4

Through these examples we have seen how, by studying the phase plane, one can often anticipate some of the features (boundedness, periodicity, etc.) of solutions of autonomous systems without solving them explicitly. Much of this information can be predicted simply from the critical points and the direction field (oriented by arrowheads), which are obtainable through standard software packages. The final example ties together several of these ideas.

Example 5 Find the critical points and solve the phase plane equation (2) for

$$\frac{dx}{dt} = -y(y-2)\ ,\qquad \frac{dy}{dt} = (x-2)(y-2)\ . \tag{14}$$

What is the asymptotic behavior of the solutions starting from $(3, 0)$, $(5, 0)$, and $(2, 3)$?

Solution To find the critical points, we solve the system

$$-y(y-2) = 0, \qquad (x-2)(y-2) = 0\ .$$

One family of solutions to this system is given by $y = 2$ with x arbitrary; that is, the line $y = 2$. If $y \neq 2$, then the system simplifies to $-y = 0$, and $x - 2 = 0$, which has the solution $x = 2$, $y = 0$. Hence, the critical point set consists of the isolated point $(2, 0)$ and the horizontal line $y = 2$. The corresponding equilibrium solutions are $x(t) \equiv 2$, $y(t) \equiv 0$, and the family $x(t) \equiv c$, $y(t) \equiv 2$, where c is an arbitrary constant.

The trajectories in the phase plane satisfy the equation

$$\frac{dy}{dx} = \frac{(x-2)(y-2)}{-y(y-2)} = -\frac{x-2}{y}\ . \tag{15}$$

Solving (15) by separating variables,

$$y\,dy = -(x-2)\,dx \qquad \text{or} \qquad y^2 + (x-2)^2 = C\ ,$$

demonstrates that the trajectories lie on concentric circles centered at $(2, 0)$. See Figure 5.15.

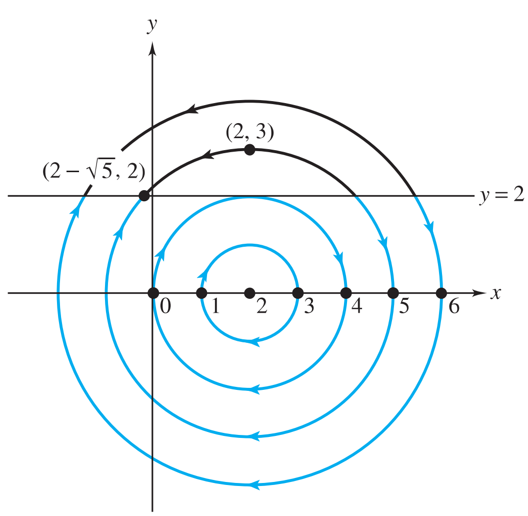

Figure 5.15 Phase plane diagram for Example 5

Next we analyze the flow along each trajectory. From the equation $dx/dt = -y(y-2)$, we see that x is decreasing when $y > 2$. This means the flow is from right to left along the arc of a circle that lies above the line $y = 2$. For $0 < y < 2$, we have $dx/dt > 0$, so in this region the flow is from left to right. Furthermore, for $y < 0$, we have $dx/dt < 0$, and again the flow is from right to left.

We now observe in Figure 5.15 that there are four types of trajectories associated with system (14): (a) those that begin above the line $y = 2$ and follow the arc of a circle counterclockwise back to that line; (b) those that begin below the line $y = 2$ and follow the arc of a circle clockwise back to that line; (c) those that continually move clockwise around a circle centered at $(2, 0)$ with radius less than 2 (i.e., they do not intersect the line $y = 2$); and finally, (d) the critical points $(2, 0)$ and $y = 2$, x arbitrary.

The solution starting at $(3, 0)$ lies on a circle with no critical points; therefore, it is a *periodic* solution, and the critical point $(2, 0)$ is a center. But the circle containing the solutions starting at $(5, 0)$ and at $(2, 3)$ has critical points at $\left(2 - \sqrt{5}, 2\right)$ and $\left(2 + \sqrt{5}, 2\right)$. The direction arrows indicate that both solutions approach $\left(2 - \sqrt{5}, 2\right)$ asymptotically (as $t \to +\infty$). They lie on the same circle (or phase plane solution curve), but they are quite different trajectories. ◆

Note that for the system (14) the critical points on the line $y = 2$ are not isolated, so they do not fit into any of the categories depicted in Figure 5.12. Observe also that all solutions of this system are bounded, since they are confined to circles.

5.4 EXERCISES

In Problems 1 and 2, verify that the pair $x(t)$, $y(t)$ is a solution to the given system. Sketch the trajectory of the given solution in the phase plane.

1. $\dfrac{dx}{dt} = 3y^3 , \quad \dfrac{dy}{dt} = y ;$

$x(t) = e^{3t} , \quad y(t) = e^t$

2. $\dfrac{dx}{dt} = 1 , \quad \dfrac{dy}{dt} = 3x^2 ;$

$x(t) = t + 1, \quad y(t) = t^3 + 3t^2 + 3t$

In Problems 3–6, find the critical point set for the given system.

3. $\dfrac{dx}{dt} = x - y ,$

$\dfrac{dy}{dt} = x^2 + y^2 - 1$

4. $\dfrac{dx}{dt} = y - 1 ,$

$\dfrac{dy}{dt} = x + y + 5$

5. $\dfrac{dx}{dt} = x^2 - 2xy ,$

$\dfrac{dy}{dt} = 3xy - y^2$

6. $\dfrac{dx}{dt} = y^2 - 3y + 2 ,$

$\dfrac{dy}{dt} = (x - 1)(y - 2)$

In Problems 7–9, solve the phase plane equation (2), page 264, for the given system.

7. $\dfrac{dx}{dt} = y - 1 ,$

$\dfrac{dy}{dt} = e^{x+y}$

8. $\dfrac{dx}{dt} = x^2 - 2y^{-3} ,$

$\dfrac{dy}{dt} = 3x^2 - 2xy$

9. $\dfrac{dx}{dt} = 2y - x ,$

$\dfrac{dy}{dt} = e^x + y$

10. Find all the critical points of the system

$$\frac{dx}{dt} = x^2 - 1 ,$$
$$\frac{dy}{dt} = xy ,$$

and the xy-phase plane solution curves. Thereby prove that there are two trajectories that are genuine semicircles. What are the endpoints of the semicircles?

In Problems 11–14, solve the phase plane equation for the given system. Then sketch by hand several representative trajectories (with their flow arrows).

11. $\dfrac{dx}{dt} = 2y\,,\quad \dfrac{dy}{dt} = 2x$

12. $\dfrac{dx}{dt} = -8y\,,\quad \dfrac{dy}{dt} = 18x$

13. $\dfrac{dx}{dt} = (y - x)(y - 1)\,,\quad \dfrac{dy}{dt} = (x - y)(x - 1)$

14. $\dfrac{dx}{dt} = \dfrac{3}{y}\,,\quad \dfrac{dy}{dt} = \dfrac{2}{x}$

In Problems 15–18, find all critical points for the given system. Then use a software package to sketch the direction field in the phase plane and from this describe the stability of the critical points (i.e., compare with Figure 5.12).

15. $\dfrac{dx}{dt} = 2x + y + 3\,,\quad \dfrac{dy}{dt} = -3x - 2y - 4$

16. $\dfrac{dx}{dt} = -5x + 2y\,,\quad \dfrac{dy}{dt} = x - 4y$

17. $\dfrac{dx}{dt} = 2x + 13y\,,\quad \dfrac{dy}{dt} = -x - 2y$

18. $\dfrac{dx}{dt} = x(7 - x - 2y)\,,\quad \dfrac{dy}{dt} = y(5 - x - y)$

In Problems 19–24, convert the given second-order equation into a first-order system by setting $v = y'$. Then find all the critical points in the yv-plane. Finally, sketch (by hand or software) the direction fields, and describe the stability of the critical points (i.e., compare with Figure 5.12).

19. $\dfrac{d^2y}{dt^2} - y = 0$

20. $\dfrac{d^2y}{dt^2} + y = 0$

21. $\dfrac{d^2y}{dt^2} + y + y^5 = 0$

22. $\dfrac{d^2y}{dt^2} + y^3 = 0$

23. $y''(t) + y(t) - y(t)^4 = 0$

24. $y''(t) + y(t) - y(t)^3 = 0$

25. Using software, sketch the direction field in the phase plane for the system

$$dx/dt = y\,,$$
$$dy/dt = -x + x^3\,.$$

From the sketch, conjecture whether the solution passing through each given point is periodic:

(a) $(0.25, 0.25)$ **(b)** $(2, 2)$ **(c)** $(1, 0)$

26. Using software, sketch the direction field in the phase plane for the system

$$dx/dt = y\,,$$
$$dy/dt = -x - x^3\,.$$

From the sketch, conjecture whether all solutions of this system are bounded. Solve the phase plane equation and confirm your conjecture.

27. Using software, sketch the direction field in the phase plane for the system

$$dx/dt = -2x + y\,,$$
$$dy/dt = -5x - 4y\,.$$

From the sketch, predict the asymptotic limit (as $t \to +\infty$) of the solution starting at $(1, 1)$.

28. Figure 5.16 displays some trajectories for the system

$$dx/dt = y\,,$$
$$dy/dt = -x + x^2\,.$$

What types of critical points (compare Figure 5.12) occur at $(0, 0)$ and $(1, 0)$?

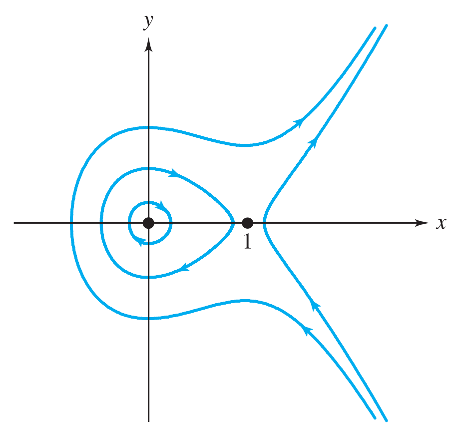

Figure 5.16 Phase plane for Problem 28

29. The phase plane diagrams depicted in Figure 5.12 were derived from the following systems. Use any method (except software) to match the systems to the graphs.

(a) $dx/dt = x\,,\quad dy/dt = 3y$

(b) $dx/dt = y/2\,,\quad dy/dt = -2x$

(c) $dx/dt = -5x + 2y\,,\quad dy/dt = x - 4y$

(d) $dx/dt = 2x - y\,,\quad dy/dt = x + 2y$

(e) $dx/dt = 5x - 3y\,,\quad dy/dt = 4x - 3y$

(f) $dx/dt = -y\,,\quad dy/dt = x - y$

30. A proof of Theorem 1, page 268, is outlined below. The goal is to show that $f(x^*, y^*) = g(x^*, y^*) = 0$. Justify each step.

(a) From the given hypotheses, deduce that $\lim_{t\to+\infty} x'(t) = f(x^*, y^*)$ and $\lim_{t\to+\infty} y'(t) = g(x^*, y^*)$.

(b) Suppose $f(x^*, y^*) > 0$. Then, by continuity, $x'(t) > f(x^*, y^*)/2$ for all large t (say, for $t \geq T$). Deduce from this that $x(t) > tf(x^*, y^*)/2 + C$ for $t > T$, where C is some constant.

(c) Conclude from part (b) that $\lim_{t\to+\infty} x(t) = +\infty$, contradicting the fact that this limit is the finite number x^*. Thus, $f(x^*, y^*)$ cannot be positive.

(d) Argue similarly that the supposition that $f(x^*, y^*) < 0$ also leads to a contradiction; hence, $f(x^*, y^*)$ must be zero.

(e) In the same manner, argue that $g(x^*, y^*)$ must be zero.

Therefore, $f(x^*, y^*) = g(x^*, y^*) = 0$, and (x^*, y^*) is a critical point.

31. Phase plane analysis provides a quick derivation of the energy integral lemma of Section 4.8 (page 204). By completing the following steps, prove that solutions of equations of the special form $y'' = f(y)$ satisfy

$$\frac{1}{2}(y')^2 - F(y) = \text{constant} ,$$

where $F(y)$ is an antiderivative of $f(y)$.

(a) Set $v = y'$ and write $y'' = f(y)$ as an equivalent first-order system.

(b) Show that the solutions to the yv-phase plane equation for the system in part (a) satisfy $v^2/2 = F(y) + K$. Replacing v by y' then completes the proof.

32. Use the result of Problem 31 to prove that all solutions to the equation

$$y'' + y^3 = 0$$

remain bounded. [*Hint:* Argue that $y^4/4$ is bounded above by the constant appearing in Problem 31.]

33. A Problem of Current Interest. The motion of an iron bar attracted by the magnetic field produced by a parallel current wire and restrained by springs (see Figure 5.17) is governed by the equation

$$\frac{d^2x}{dt^2} = -x + \frac{1}{\lambda - x} , \quad \text{for} \quad -x_0 < x < \lambda ,$$

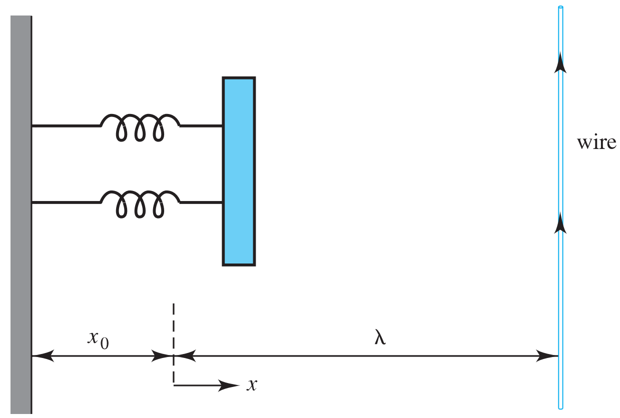

Figure 5.17 Bar restrained by springs and attracted by a parallel current

where the constants x_0 and λ are, respectively, the distances from the bar to the wall and to the wire *when the bar is at equilibrium (rest) with the current off.*

(a) Setting $v = dx/dt$, convert the second-order equation to an equivalent first-order system.

(b) Solve the phase plane equation for the system in part (a) and thereby show that its solutions are given by

$$v = \pm\sqrt{C - x^2 - 2\ln(\lambda - x)} ,$$

where C is a constant.

(c) Show that if $\lambda < 2$ there are no critical points in the xv-phase plane, whereas if $\lambda > 2$ there are two critical points. For the latter case, determine these critical points.

(d) Physically, the case $\lambda < 2$ corresponds to a current so high that the magnetic attraction completely overpowers the spring. To gain insight into this, use software to plot the phase plane diagrams for the system when $\lambda = 1$ and when $\lambda = 3$.

(e) From your phase plane diagrams in part (d), describe the possible motions of the bar when $\lambda = 1$ and when $\lambda = 3$, under various initial conditions.

34. Falling Object. The motion of an object moving vertically through the air is governed by the equation

$$\frac{d^2y}{dt^2} = -g - \frac{g}{V^2}\frac{dy}{dt}\left|\frac{dy}{dt}\right| ,$$

where y is the upward vertical displacement and V is a constant called the terminal speed. Take $g = 32$ ft/sec^2 and $V = 50$ ft/sec. Sketch trajectories in the yv-phase plane for $-100 \le y \le 100$, $-100 \le v \le 100$, starting from $y = 0$ and $v = -75, -50, -25, 0, 25, 50$, and 75 ft/sec. Interpret the trajectories physically; why is V called the terminal speed?

35. Sticky Friction. An alternative for the damping friction model $F = -by'$ discussed in Section 4.1 is the "sticky friction" model. For a mass sliding on a surface as depicted in Figure 5.18, the contact friction is more complicated than simply $-by'$. We observe, for example, that even if the mass is displaced slightly off the equilibrium location $y = 0$, it may nonetheless remain stationary due to the fact that the spring force $-ky$ is insufficient to break the static friction's grip. If the maximum force that the friction can exert is denoted by μ, then a feasible model is given by

$$F_{\text{friction}} = \begin{cases} ky\ , & \text{if } |ky| < \mu \text{ and } y' = 0\ , \\ \mu\,\text{sign}(y)\ , & \text{if } |ky| \ge \mu \text{ and } y' = 0\ , \\ -\mu\,\text{sign}(y')\ , & \text{if } y' \ne 0\ . \end{cases}$$

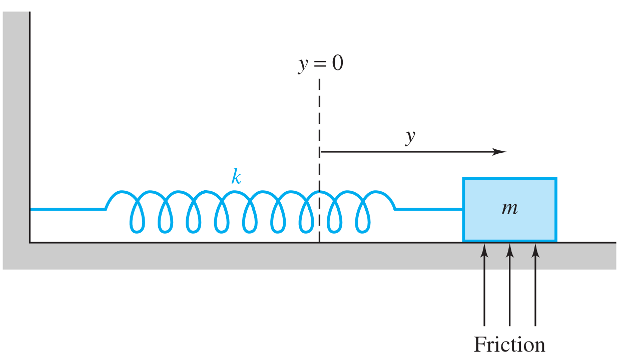

Figure 5.18 Mass–spring system with friction

(The function sign(s) is $+1$ when $s > 0$, -1 when $s < 0$, and 0 when $s = 0$.) The motion is governed by the equation

$$m\frac{d^2y}{dt^2} = -ky + F_{\text{friction}}\ . \tag{16}$$

Thus, if the mass is at rest, friction *balances* the spring force if $|y| < \mu/k$ but simply *opposes* it with intensity μ if $|y| \ge \mu/k$. If the mass is moving, friction opposes the velocity with the same intensity μ.

(a) Taking $m = \mu = k = 1$, convert (16) into the first-order system

$$\begin{aligned} y' &= v\ , \\ v' &= \begin{cases} 0\ , & \text{if } |y| < 1 \text{ and } v = 0\ , \\ -y + \text{sign}(y)\ , & \text{if } |y| \ge 1 \text{ and } v = 0\ , \\ -y - \text{sign}(v)\ , & \text{if } v \ne 0\ . \end{cases} \end{aligned} \tag{17}$$

(b) Form the phase plane equation for (17) when $v \ne 0$ and solve it to derive the solutions

$$v^2 + (y \pm 1)^2 = c\ ,$$

where the plus sign prevails for $v > 0$ and the minus sign for $v < 0$.

(c) Identify the trajectories in the phase plane as two families of concentric semicircles. What is the center of the semicircles in the upper half-plane? The lower half-plane?

(d) What are the critical points for (17)?

(e) Sketch the trajectory in the phase plane of the mass released from rest at $y = 7.5$. At what value for y does the mass come to rest?

36. Rigid Body Nutation. Euler's equations describe the motion of the principal-axis components of the angular velocity of a freely rotating rigid body (such as a space station), as seen by an observer rotating with the body (the astronauts, for example). This motion is called *nutation*. If the angular velocity components are denoted by x, y, and z, then an example of Euler's equations is the three-dimensional autonomous system

$$\begin{aligned} dx/dt &= yz\ , \\ dy/dt &= -2xz\ , \\ dz/dt &= xy\ . \end{aligned}$$

The *trajectory* of a solution $x(t), y(t), z(t)$ to these equations is the curve generated by the points $\big(x(t), y(t), z(t)\big)$ in xyz-phase space as t varies over an interval I.

(a) Show that each trajectory of this system lies on the surface of a (possibly degenerate) sphere centered at the origin $(0, 0, 0)$. [*Hint:* Compute $\frac{d}{dt}(x^2 + y^2 + z^2)$.] What does this say about the magnitude of the angular velocity vector?

(b) Find all the critical points of the system, i.e., all points (x_0, y_0, z_0) such that $x(t) \equiv x_0$, $y(t) \equiv y_0$, $z(t) \equiv z_0$ is a solution. For such solutions, the angular velocity vector remains constant in the body system.

(c) Show that the trajectories of the system lie along the intersection of a sphere and an elliptic cylinder of the form $y^2 + 2x^2 = C$, for some constant C. [*Hint:* Consider the expression for dy/dx implied by Euler's equations.]

(d) Using the results of parts (b) and (c), argue that the trajectories of this system are *closed* curves. What does this say about the corresponding solutions?

(e) Figure 5.19 displays some typical trajectories for this system. Discuss the stability of the three critical points indicated on the positive axes.

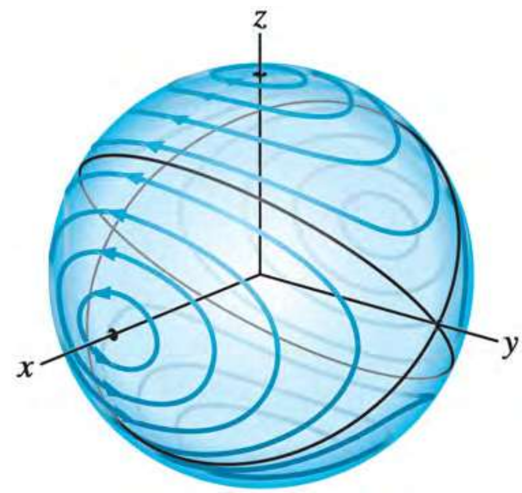

Figure 5.19 Trajectories for Euler's system

5.5 APPLICATIONS TO BIOMATHEMATICS: EPIDEMIC AND TUMOR GROWTH MODELS

In this section we are going to survey some issues in biological systems that have been successfully modeled by differential equations. We begin by reviewing the population models described in Sections 3.2 and 5.3.

In the *Malthusian model*, the rate of growth of a population $p(t)$ is proportional to the size of the existing population:

$$\frac{dp}{dt} = kp \quad (k > 0) \ . \tag{1}$$

Cells that reproduce by splitting, such as amoebae and bacteria, are obvious biological examples of this type of growth. Equation (1) implies that a Malthusian population grows exponentially; there is no mechanism for constraining the growth. In Section 3.2 we saw that certain populations exhibit Malthusian growth over limited periods of time (as does compound interest).†

Inserting a negative growth rate,

$$\frac{dp}{dt} = -kp \ , \tag{2}$$

results in solutions that *decay* exponentially. Their average lifetime is $1/k$, and their half-life is $(\ln 2)/k$ (Problems 6 and 8). In animals, certain organs such as the kidney serve to cleanse the bloodstream of unwanted components (creatinine clearance, renal clearance), and their concentrations diminish exponentially. As a general rule, the body tends to dissipate ingested drugs in such a manner. (Of course, the most familiar physical instance of Malthusian disintegration is radioactive decay.) Note that if there are both growth and extinction processes, $dp/dt = k_+p - k_-p$ and the equation in (1) still holds with $k = k_+ - k_-$.

†Gordon E. Moore (1929–) has observed that the number of transistors on new integrated circuits produced by the electronics industry doubles every 24 months. "Moore's law" is commonly cited by industrialists.

When there are two-party interactions occurring in the population that decrease the growth rate, such as competition for resources or violent crime, the *logistic model* might be applicable; it assumes that the extinction rate is proportional to the number of possible pairs in the population, $p(p-1)/2$:

(3) $$\frac{dp}{dt} = k_1 p - k_2 \frac{p(p-1)}{2} \quad \text{or, equivalently,} \quad \frac{dp}{dt} = -Ap(p-p_1)\ .^{\dagger}$$

Rodent, bird, and plant populations exhibit logistic growth rates due to social structure, territoriality, and competition for light and space, respectively. The logistic function

$$p(t) = \frac{p_0 p_1}{p_0 + (p_1 - p_0)e^{-Ap_1 t}}\ , \qquad p_0 := p(0)$$

was shown in Section 3.2 to be the solution of (3), and typical graphs of $p(t)$ were displayed there.

In Section 5.3 we observed that the *Volterra–Lotka model* for *two different* populations, a *predator* $x_2(t)$ and a *prey* $x_1(t)$, postulates a Malthusian growth rate for the prey and an extinction rate governed by $x_1 x_2$, the number of possible pairings of one from each population,

(4) $$\frac{dx_1}{dt} = Ax_1 - Bx_1x_2\ ,$$

while predators follow a Malthusian extinction rate and pairwise growth rate

(5) $$\frac{dx_2}{dt} = -Cx_2 + Dx_1x_2\ .$$

Volterra–Lotka dynamics have been observed in blood vessel growth (predator = new capillary tips; prey = chemoattractant), fish populations, and several animal–plant interactions.

Systems like (4)–(5) were studied in Section 5.3 with the aid of the Runge–Kutta algorithm. Now, armed with the insights of Section 5.4, we can further explore this model theoretically.

First, we perform a "reality check" by proving that the populations $x_1(t)$, $x_2(t)$ in the Volterra–Lotka model never change sign. Separating (4) leads to

$$\frac{1}{x_1}\frac{dx_1}{dt} = \frac{d\ln x_1}{dt} = A - Bx_2\ ,$$

while integrating from 0 to t results in

(6) $$x_1(t) = x_1(0)e^{\int_0^t \{A - Bx_2(\tau)\}d\tau}\ ,$$

and the exponential factor is always positive. Thus $x_1(t)$ [and similarly $x_2(t)$] retains its initial sign (*negative* populations never arise).

Example 1 Find and interpret the critical points for the Volterra–Lotka model (4)–(5).

Solution The system

(7) $$\begin{aligned} \frac{dx_1}{dt} &= Ax_1 - Bx_1x_2 = -Bx_1\left(x_2 - \frac{A}{B}\right) = 0\ , \\ \frac{dx_2}{dt} &= -Cx_2 + Dx_1x_2 = Dx_2\left(x_1 - \frac{C}{D}\right) = 0 \end{aligned}$$

---

† One might propose that the *growth* rate in animal populations is due to *two*-party interactions as well. (Wink, wink.) However, in monogamous societies, the number of pairs participating in procreation is proportional to $p/2$, leading to (1). A growth rate determined by *all* possible pairings $p(p-1)/2$ would indicate an extremely utopian social order.

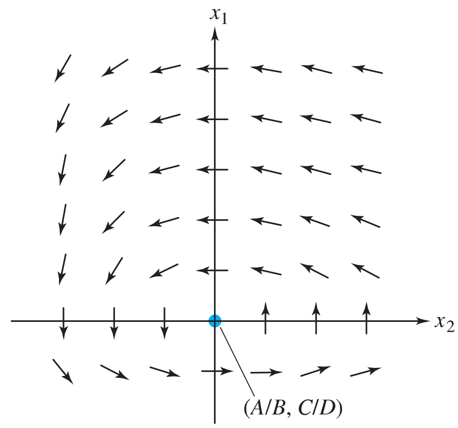

Figure 5.20 Typical direction field diagram for the Volterra–Lotka system

has the trivial solution $x_1(t) \equiv x_2(t) \equiv 0$, with an obvious interpretation in terms of populations. If all four coefficients A, B, C, and D are positive, there is also the more interesting solution

(8) $$x_2(t) \equiv \frac{A}{B}\ , \quad x_1(t) \equiv \frac{C}{D}\ .$$

At these population levels, the growth and extinction rates for each species cancel. The direction field diagram in Figure 5.20 for the phase plane equation

(9) $$\frac{dx_2}{dx_1} = \frac{-Cx_2 + Dx_1x_2}{Ax_1 - Bx_1x_2} = \frac{x_2}{x_1}\frac{-C + Dx_1}{A - Bx_2}$$

suggests that this equilibrium is a *center* (compare Figure 5.12) with closed (periodic) neighboring trajectories, in accordance with the simulations in Section 5.3. However it is conceivable that some *spiral* trajectories might snake through the field pattern and approach the critical point asymptotically. A rather tricky argument in Problem 4 demonstrates that this is *not* the case. ◆

The SIR Epidemic Model. The SIR† model for an epidemic addresses the spread of diseases that are only contracted by contact with an infected individual; its victims, once recovered, are immune to further infection and are themselves noninfectious. So the members of a population of size N fall into three classes:

$S(t)$ = the number of susceptible individuals—that is, those who have not been infected; $s := S/N$ is the fraction of susceptibles.

$I(t)$ = the number of individuals who are currently infected, comprising a fraction $i := I/N$ of the population.

$R(t)$ = the number of individuals who have recovered from infection, comprising the fraction $r := R/N$.

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†Introduced by W. O. Kermack and A. G. McKendrick in "A Contribution to the Mathematical Theory of Epidemics," *Proc. Royal Soc. London,* Vol. A115 (1927): 700–721.

The classic SIR epidemic model assumes that on the average, an infectious individual encounters a people per unit time (usually per week). Thus, a total of aI people per week are contacted by infectees, but only a fraction $s = S/N$ of them are susceptible. So the susceptible population diminishes at a rate

$$\frac{dS}{dt} = -saI \quad \text{or} \quad (\text{dividing by } N) \tag{10}$$

$$\frac{ds}{dt} = -asi \ . \tag{11}$$

The parameter a is crucial in disease control. Crowded conditions, or high a, make it difficult to combat the spread of infection. Ideally, we would quarantine the infectees (low a) to fight the epidemic.

The infected population is (obviously) increased whenever a susceptible individual is infected. Additionally, infectees recover in a Malthusian-disintegrative manner over an average time of, say, $1/k$ weeks [recall (1)], so the infected population changes at a rate

$$\frac{dI}{dt} = saI - kI = a\left(s - \frac{k}{a}\right)I \quad \text{or} \tag{12}$$

$$\frac{di}{dt} = a\left(s - \frac{k}{a}\right)i \ . \tag{13}$$

And, of course, the population of recovered individuals increases whenever an infectee is healed:

$$\frac{dR}{dt} = kI \quad \text{or} \quad \frac{dr}{dt} = ki \ . \tag{14}$$

With the SIR model, the total population count remains unchanged:

$$\frac{d(S + I + R)}{dt} = -saI + saI - kI + kI = 0 \ .$$

Thus, any fatalities are tallied in the "recovered/noninfectious" population R.

Interestingly, equations (11) and (13) do not contain R or r, so they are suitable for phase plane analysis. In fact they constitute a Volterra–Lotka system with $A = 0$, $B = D = a$, and $C = k$. Because the coefficient A is zero, the critical point structure is different from that discussed in Example 1. Specifically, if $-asi$ in (11) is zero, then only $-ki$ remains on the right in (13), so $I(t) = i(t) \equiv 0$ is necessary and sufficient for a critical point, with S unrestricted. (Physically, this means the populations remain stable only if there are no carriers of the infection.)

Our earlier argument has shown that if $s(t)$ and $i(t)$ are initially positive, they remain so. As a result we conclude from (11) immediately that $s(t)$ decreases monotonically; as such, it has a limiting value $s(\infty)$ as $t \to \infty$. Does $i(t)$ have a limiting value also? If so, $\{s(\infty), i(\infty)\}$ would be a critical point by Theorem 1 of Section 5.4, and thus $i(\infty) = 0$.

To analyze $i(t)$ consider the phase plane equation for (11) and (13):

$$\frac{di}{ds} = \frac{asi - ki}{-asi} = -1 + \frac{k}{as} \ , \tag{15}$$

which has solutions

$$i = -s + \frac{k}{a}\ln s + K \ . \tag{16}$$

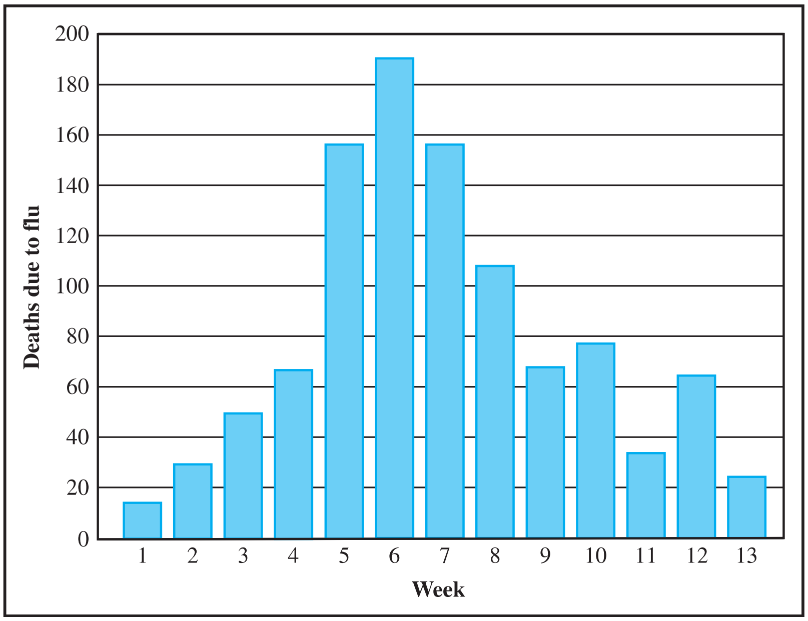

Figure 5.21 Mortality data for Hong Kong flu, New York City

From (16) we see that $s(\infty)$ cannot be zero; otherwise the right-hand side would eventually be negative, contradicting $i(t) > 0$. Therefore, (16) demonstrates that $i(t)$ *does* have a limiting value $i(\infty) = -s(\infty) + (k/a)\ln s(\infty) + K$. As noted, $i(\infty)$ must then be zero.

From (13) we further conclude that if $s(0)$ exceeds the "threshold value" k/a, the infected fraction $i(t)$ will initially increase ($di/dt > 0$ at $t = 0$) before eventually dying out. The peak value of $i(t)$ occurs when $di/dt = 0 = a[s - (k/a)]i$, i.e., when $s(t)$ passes through the value k/a. In the jargon of epidemiology, this phenomenon defines an "epidemic." You will be directed in Problem 10 to show that if $s(0) \le k/a$, the infected population diminishes monotonically, and no epidemic develops.

Example 2† According to data issued by the Centers for Disease Control and Prevention (CDC) in Atlanta, Georgia, the Hong Kong flu epidemic during the winter of 1968–1969 was responsible for 1035 deaths in New York City (population 7,900,000), according to the time chart in Figure 5.21. Analyze this data with the SIR model.

Solution Of course, we need to make some assumptions about the parameters. First of all, only a small percentage of people who contract Hong Kong flu perish, so let's assume that the chart reflects a scaled version of the infected population fraction $i(t)$. It is known that the recovery period for this flu is around 5 days, or 5/7 week, so we try $k = 7/5 = 1.4$. And since the infectees spend much of their convalescence in bed, the average contact rate a is probably less than 1 person per day or 7 per week. The CDC estimated that the initial infected population $I(0)$ was about 10, so the initial data for (11), (13), and (14) are

$$s(0) = \frac{7{,}900{,}000 - 10}{7{,}900{,}000} \approx 0.9999987, \quad i(0) = \frac{10}{7{,}900{,}000} \approx 1.2658 \times 10^{-6}, \quad r(0) = 0 \ .$$

†We borrow liberally from "The SIR Model for Spread of Disease" by Duke University's David Smith and Lang Moore, *Journal of Online Mathematics and Its Applications,* The MAA Mathematical Sciences Digital Library, http://mathdl.maa.org/mathDL/4/?pa=content&sa=viewDocument&nodeId=479&bodyId=612, copyright 2000, CCP and the authors, published December, 2001. The article contains much interesting information about this epidemic.

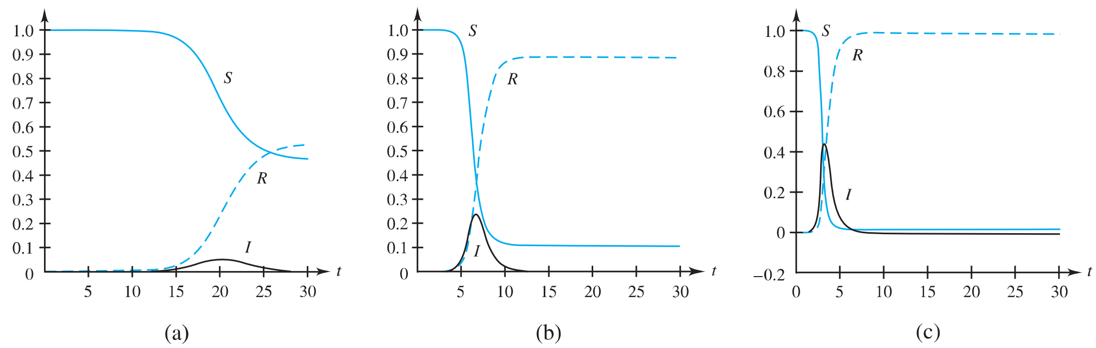

Figure 5.22 SIR simulations (a) $k = 1.4$, $a = 2.0$; (b) $k = 1.4$, $a = 3.5$; (c) $k = 1.4$, $a = 6.0$

Numerical simulations of this system are displayed in Figure 5.22.† The contact rate $a = 3.5$ per week generates an infection fraction curve that closely matches the mortality data's characteristics: time of peak and duration of epidemic. ◆

A Tumor Growth Model.†† The observed growth of certain tumors can be explained by a model that is mathematically similar to the epidemic model. The total number of cells N in the tumor subdivides into a population P that proliferates by splitting (Malthusian growth) and a population Q that remains quiescent. However, the proliferating cells also can make a transition to the quiescent state, and this occurrence is modeled as a Malthusian-like decay with a "rate" $r(N)$ *that increases with the overall size of the tumor:*

$$\textbf{(17)} \qquad \frac{dP}{dt} = cP - r(N)P \ ,$$

$$\textbf{(18)} \qquad \frac{dQ}{dt} = r(N)P \ .$$

Thus the total population N increases only when the proliferating cells split, as can be seen by adding the equations (17) and (18):

$$\textbf{(19)} \qquad \frac{dN}{dt} = \frac{d(P + Q)}{dt} = cP \ .$$

We take (17) and (19) as the system for our analysis. The phase plane equation

$$\textbf{(20)} \qquad \frac{dP}{dN} = \frac{cP - r(N)P}{cP} = 1 - \frac{r(N)}{c}$$

can be integrated, leading to a formula for P in terms of N

$$\textbf{(21)} \qquad P = N - \frac{1}{c}\int r(N)\, dN + K \ .$$

†An applet, maintained on the Web at http://alamos.math.arizona.edu/~rychlik/JOde/index.html, automates most of the differential equation algorithms discussed in this book.

††The authors wish to thank Dr. Glenn Webb of Vanderbilt University for this application. See M. Gyllenberg and G. F. Webb, "Quiescence as an Explanation of Gompertz Tumor Growth," *Growth, Development, and Aging,* Vol. 53 (1989): 25–55.

Suppose the initial conditions are $P(0) = 1$, $Q(0) = 0$, and $N(0) = 1$ (a single proliferating cell). Then we can eliminate the nuisance constant K by taking the indefinite integral in (21) to run from 1 to N and evaluating at $t = 0$:

$$1 = 1 - \frac{1}{c}\int_1^1 r(N)\,dN + K \quad \Rightarrow \quad K = 0 \ .$$

Insertion of (21) with $K = 0$ into (19) produces a differential equation for N alone:

$$\frac{dN}{dt} = cN - \int_1^N r(u)\,du \ . \tag{22}$$

Example 3 The *Gompertz law*

$$N(t) = e^{c(1-e^{-bt})/b} \tag{23}$$

has been observed experimentally for the growth of some tumors. Show that a transition rate $r(N)$ of the form $b(1 + \ln N)$ predicts Gompertzian growth.

Solution If the indicated integral of the rate $r(N)$ is carried out, (22) becomes

$$\frac{dN}{dt} = cN - b(N-1) - b(N\ln N - N + 1) = (c - b\ln N)\,N \ . \tag{24}$$

Dividing by N we obtain a linear differential equation for the function $\ln N$

$$\frac{d\ln N}{dt} = -b\ln N + c$$

whose solution, for the initial condition $N(0) = 1$, is found by the methods of Section 2.3 to be

$$\ln N(t) = \frac{c}{b}(1 - e^{-bt}) \ ,$$

confirming (23). ◆

Problem 9 invites the reader to show that if the growth rate is modeled as $r(N) = s(2N - 1)$, then the solution of (22) describes logistic growth. Typical curves for the Gompertz and logistic models are displayed in Figure 5.23. See also Figure 3.4 on page 97.

Other applications of differential equations to biomathematics appear in the discussions of artificial respiration (Project B) in Chapter 2, HIV infection (Project A) and aquaculture (Project B) in Chapter 3, and spread of staph infections (Project B) and growth of phytoplankton (Project F) in this chapter.

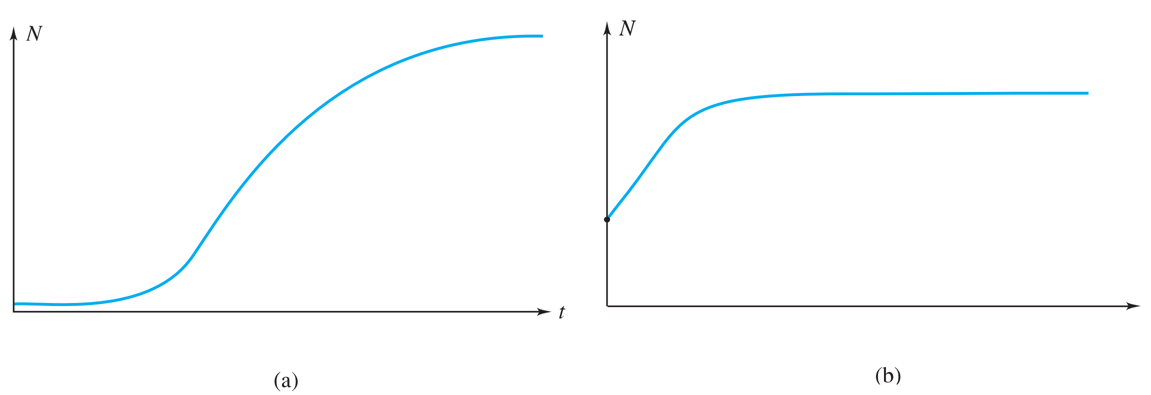

Figure 5.23 (a) Gompertz and (b) logistic curves

5.5 EXERCISES

1. Logistic Model. In Section 3.2 we discussed the logistic equation

$$\frac{dp}{dt} = Ap_1p - Ap^2 , \qquad p(0) = p_0 ,$$

and its use in modeling population growth. A more general model might involve the equation

$$\frac{dp}{dt} = Ap_1p - Ap^r , \qquad p(0) = p_0 , \tag{25}$$

where $r > 1$. To see the effect of changing the parameter r in (25), take $p_1 = 3$, $A = 1$, and $p_0 = 1$. Then use a numerical scheme such as Runge–Kutta with $h = 0.25$ to approximate the solution to (25) on the interval $0 \le t \le 5$ for $r = 1.5$, 2, and 3. What is the limiting population in each case? For $r > 1$, determine a general formula for the limiting population.

2. Radioisotopes and Cancer Detection. A radioisotope commonly used in the detection of breast cancer is technetium-99m. This radionuclide is attached to a chemical that upon injection into a patient accumulates at cancer sites. The isotope's radiation is then detected and the site located, using gamma cameras or other tomographic devices.

Technetium-99m decays radioactively in accordance with the equation $dy/dt = -ky$, with $k = 0.1155$/h. The short half-life of technetium-99m has the advantage that its radioactivity does not endanger the patient. A disadvantage is that the isotope must be manufactured in a cyclotron. Since hospitals are not equipped with cyclotrons, doses of technetium-99m have to be ordered in advance from medical suppliers.

Suppose a dosage of 5 millicuries (mCi) of technetium-99m is to be administered to a patient. Estimate the delivery time from production at the manufacturer to arrival at the hospital treatment room to be 24 h and calculate the amount of the radionuclide that the hospital must order, to be able to administer the proper dosage.

3. Secretion of Hormones. The secretion of hormones into the blood is often a periodic activity. If a hormone is secreted on a 24-h cycle, then the rate of change of the level of the hormone in the blood may be represented by the initial value problem

$$\frac{dx}{dt} = \alpha - \beta\cos\frac{\pi t}{12} - kx , \quad x(0) = x_0 ,$$

where $x(t)$ is the amount of the hormone in the blood at time t, α is the average secretion rate, β is the amount of daily variation in the secretion, and k is a positive constant reflecting the rate at which the body removes the hormone from the blood. If $\alpha = \beta = 1$, $k = 2$, and $x_0 = 10$, solve for $x(t)$.

4. Prove that the critical point (8) of the Volterra–Lotka system is a center; that is, the neighboring trajectories are periodic. *Hint:* Observe that (9) is separable and show that its solutions can be expressed as

$$[x_2^A e^{-Bx_2}] \cdot [x_1^C e^{-Dx_1}] = K . \tag{26}$$

Prove that the maximum of the function $x^p e^{-qx}$ is $(p/qe)^p$, occurring at the unique value $x = p/q$ (see Figure 5.24), so the critical values (8) maximize the factors on the left in (26). Argue that if K takes the corresponding maximum value $(A/Be)^A(C/De)^C$, the critical point (8) is the (unique) solution of (26), and it cannot be an endpoint of any trajectory for (26) with a lower value of K.[†]

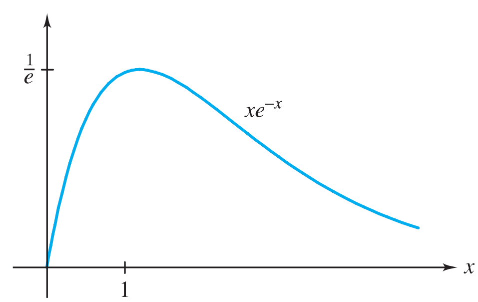

Figure 5.24 Graph of xe^{-x}

5. Suppose for a certain disease described by the SIR model it is determined that $a = 0.003$ and $b = 0.5$.

(a) In the *SI*-phase plane, sketch the trajectory corresponding to the initial condition that one person is infected and 700 persons are susceptible.

[†]In fact, the periodic fluctuations predicted by the Volterra–Lotka model were observed in fish populations by Lotka's son-in-law, Humberto D'Ancona.

(b) From your graph in part (a), estimate the peak number of infected persons. Compare this with the theoretical prediction $S = k/a \approx 167$ persons when the epidemic is at its peak.

6. Show that the half-life of solutions to (2)—that is, the time required for the solution to decay to one-half of its value—equals $(\ln 2)/k$.

7. Complete the solution of the tumor growth model for Example 3 by finding $P(t)$ and $Q(t)$.

8. If $p(t)$ is a Malthusian population that diminishes according to (2), then $p(t_2) - p(t_1)$ is the number of individuals in the population whose lifetime lies between t_1 and t_2. Argue that the *average* lifetime of the population is given by the formula

$$\frac{\displaystyle\int_0^\infty t\left|\frac{dp(t)}{dt}\right| dt}{\displaystyle\int_0^\infty p(t)\, dt}$$

and show that this equals $1/k$.

9. Show that with the transition rate formula $r(N) = s(2N - 1)$, equation (22) takes the form of the equation for the logistic model (Section 3.2, equation (14)). Solve (22) for this case.

10. Prove that the infected population $I(t)$ in the SIR model does not increase if $S(0)$ is less than or equal to k/a.

11. An epidemic reported by the British Communicable Disease Surveillance Center in the *British Medical Journal* (March 4, 1978, p. 587) took place in a boarding school with 763 residents.[†] The statistics for the infected population are shown in the graph in Figure 5.25.

Assuming that the average duration of the infection is 2 days, use a numerical differential equation solver such as the Web-based one described in Example 2 to try to reproduce the data. Take $S(0) = 762$, $I(0) = 1$, $R(0) = 0$ as initial conditions. Experiment with reasonable estimates for the average number of contacts per day by the infected students, who were confined to bed after the infection was detected. What value of this parameter seems to fit the curve best?

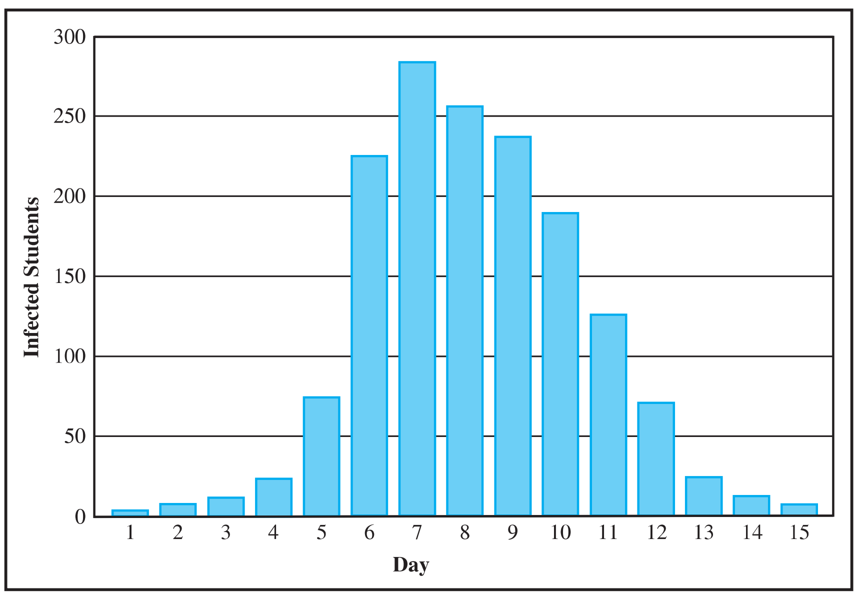

Figure 5.25 Flu data for Problem 11

[†]See also the discussion of this epidemic in *Mathematical Biology I, An Introduction,* by J. D. Murray (Springer-Verlag, New York, 2002), 325–326.